Today, we will:

- Do more examples – the stream function.
- Discuss the differential equation for momentum in fluid flow: **The Navier-Stokes eq**.
- Do some example problems – Navier-Stokes equation

Example: Streamlines in Cylindrical Coordinates

Given: A flow field is steady and 2-D in the r-θ plane, and its stream function is given by

$$\psi = V_\infty r \sin\theta$$

To do:

(**a**) Derive expressions for u_r and u_θ and convert to u and v.
(**b**) Sketch the streamlines for this flow field.

Solution:

Example: Streamlines in Cylindrical Coordinates

Given: A flow field is steady and 2-D in the r-θ plane, and its velocity field is given by

$$u_r = \frac{c}{r} \qquad u_\theta = 0 \qquad u_z = 0$$

To do: Generate an expression for stream function $\psi(r,\theta)$ and plot some streamlines.

Solution:

Derivation of the Navier-Stokes Equation
(Section 9-5, Çengel and Cimbala)

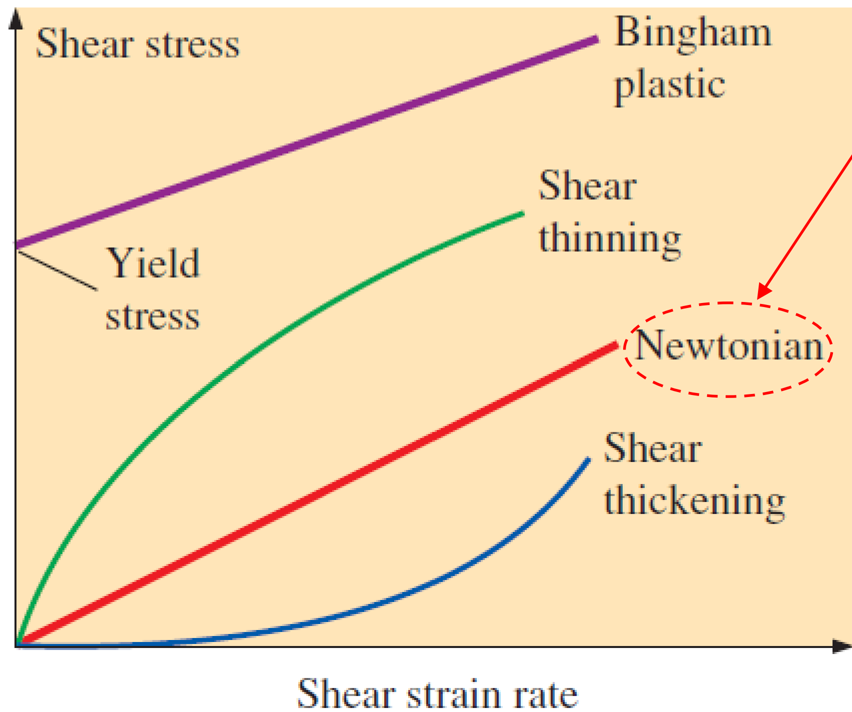
We begin with the general differential equation for conservation of linear momentum, i.e., ***Cauchy's equation***, which is valid for any kind of fluid,

Stress tensor

Cauchy's equation: $$\rho\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}\right] = \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} + \vec{\nabla}\cdot\sigma_{ij}$$

The problem is that the stress tensor σ_{ij} needs to be written in terms of the primary unknowns in the problem in order for Cauchy's equation to be useful to us. The equations that relate σ_{ij} to other variables in the problem – velocity, pressure, and fluid properties – are called ***constitutive equations***. There are different constitutive equations for different kinds of fluids.

Types of fluids:



For ***Newtonian fluids***, *the shear stress is linearly proportional to the shear strain rate.*

Examples of Newtonian fluids: water, air, oil, gasoline, most other common fluids.

Some examples of non-Newtonian fluids:
- Paint (***shear thinning*** or ***pseudo-plastic***)
- Toothpaste (***Bingham plastic***)
- Quicksand (***shear thickening*** or ***dilatant***).

We consider only Newtonian fluids in this course.

For ***Newtonian fluids*** (see text for derivation), it turns out that

$$\sigma_{ij} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} + \begin{pmatrix} 2\mu \frac{\partial u}{\partial x} & \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & 2\mu \frac{\partial v}{\partial y} & \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & 2\mu \frac{\partial w}{\partial z} \end{pmatrix} \quad \textbf{(9–57)}$$

We have achieved our goal of writing σ_{ij} in terms of pressure P, velocity components u, v , and w, and fluid viscosity μ.

Now we plug this expression for the stress tensor σ_{ij} into Cauchy's equation. The result is the famous ***Navier-Stokes equation***, shown here for incompressible flow.

Incompressible Navier–Stokes equation:

Navier-Stokes equation:

$$\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla}P + \rho\vec{g} + \mu\nabla^2\vec{V} \quad \textbf{(9–60)}$$

To solve fluid flow problems, we need both the continuity equation and the Navier-Stokes equation. Since it is a vector equation, the Navier-Stokes equation is usually split into three components in order to solve fluid flow problems. In Cartesian coordinates,

Incompressible continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \textbf{(9–61a)}$$

x-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad \textbf{(9–61b)}$$

y-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \quad \textbf{(9–61c)}$$

z-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \quad \textbf{(9–61d)}$$

2. Applications of the N-S Equation and Examples

Two main applications:

a. Determine the pressure field for a known velocity field
b. Solve fluid flow problems

Example: Stream function and how to plot streamlines

Given: A steady, 2-D, incompressible flow is given by this velocity field:

$$u = x^2 - y^2$$

$$v = -2xy$$

$$w = 0$$

There is no gravity in the *x* or *y* directions (gravity acts only in the *z* direction).

To do: Generate an expression for pressure $P(x,y)$ [the pressure field].

Solution:

First, it is wise to verify that this is a vaild steady, incompressible velocity field. If not, the problem is ill-posed and we could not continue. To verify, it must satisfy continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Now plug this velocity field into the Navier-Stokes equation.

$$\rho \frac{D\vec{V}}{Dt} = \rho\left[\frac{\partial \vec{V}}{\partial t} + \left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}\right] = -\vec{\nabla}P + \rho\vec{g} + \mu\nabla^2\vec{V}$$

Let's work on the *x*-component first:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

Now let's work on the *y*-component:

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$