Today, we will:

- Continue Chapter 10 – Approximate solutions of the N-S equation
- Show how to nondimensionalize the N-S equation
- Discuss creeping flow (flow at very low Reynolds number)

B. Nondimensionalization of the Equations of Motion (continued)

Last lecture, we derived the nondimensional form of the continuity equation,

$$\boxed{\vec{\nabla}^* \cdot \vec{V}^* = 0}$$

Now let's do the same thing with the Navier-Stokes equation.

We begin with the differential equation for conservation of linear momentum for a Newtonian fluid, i.e., the ***Navier-Stokes equation***. For incompressible flow,

$$\rho \frac{D\vec{V}}{Dt} = \rho\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}\right] = -\vec{\nabla}P + \rho\vec{g} + \mu\nabla^2\vec{V} \qquad \textbf{(10–2)}$$

Equation 10-2 is *dimensional*, and each variable or property (ρ, $\vec{V}$, t, μ, etc.) is also *dimensional*. What are the primary dimensions (in terms of {m}, {L}, {t}, {T}, etc) of each term in this equation?

Answer: $\{ \quad \}$

To ***nondimensionalize*** Eq. 10-2, we choose ***scaling parameters*** as follows:

TABLE 10–1

Scaling parameters used to nondimensionalize the continuity and momentum equations, along with their primary dimensions

Scaling Parameter	Description	Primary Dimensions
L	Characteristic length	{L}
V	Characteristic speed	$\{Lt^{-1}\}$
f	Characteristic frequency	$\{t^{-1}\}$
$P_0 - P_\infty$	Reference pressure difference	$\{mL^{-1}t^{-2}\}$
g	Gravitational acceleration	$\{Lt^{-2}\}$

We define ***nondimensional variables***, using the scaling parameters in Table 10-1:

$$t^* = ft \qquad \vec{x}^* = \frac{\vec{x}}{L} \qquad \vec{V}^* = \frac{\vec{V}}{V}$$

$$P^* = \frac{P - P_\infty}{P_0 - P_\infty} \qquad \vec{g}^* = \frac{\vec{g}}{g} \qquad \vec{\nabla}^* = L\vec{\nabla} \qquad (10\text{-}3)$$

To plug Eqs. 10-3 into Eq. 10-2, we need to first rearrange the equations in terms of the dimensional variables, i.e.,

$$t = \frac{1}{f}t^* \qquad \vec{x} = L\vec{x}^* \qquad \vec{V} = V\vec{V}^*$$

$$P = P_\infty + \left(P_0 - P_\infty\right)P^* \qquad \vec{g} = g\vec{g}^* \qquad \vec{\nabla} = \frac{1}{L}\vec{\nabla}^*$$

Now we substitute all of the above into Eq. 10-2 to obtain

$$\rho V f \frac{\partial \vec{V}^*}{\partial t^*} + \frac{\rho V^2}{L}\left(\vec{V}^*\cdot\vec{\nabla}^*\right)\vec{V}^* = -\frac{P_0 - P_\infty}{L}\vec{\nabla}^* P^* + \rho g \vec{g}^* + \frac{\mu V}{L^2}\nabla^{*2}\vec{V}^*$$

Every additive term in the above equation has primary dimensions {m^{1}L^{-2}t^{-2}}. To nondimensionalize the equation, we multiply every term by constant $L/(\rho V^2)$, which has primary dimensions {m^{-1}L^{2}t^2}, so that the dimensions cancel. After some rearrangement,

$$\boxed{\left[\frac{fL}{V}\right]}\frac{\partial \vec{V}^*}{\partial t^*} + \left(\vec{V}^*\cdot\vec{\nabla}^*\right)\vec{V}^* = -\boxed{\left[\frac{P_0 - P_\infty}{\rho V^2}\right]}\vec{\nabla}^* P^* + \boxed{\left[\frac{gL}{V^2}\right]}\vec{g}^* + \boxed{\left[\frac{\mu}{\rho V L}\right]}\nabla^{*2}\vec{V}^* \qquad \textbf{(10–5)}$$

Strouhal number, where $\mathrm{St} = \frac{fL}{V}$

Euler number, where $\mathrm{Eu} = \frac{P_0 - P_\infty}{\rho V^2}$

Inverse of Froude number squared, where $\mathrm{Fr} = \frac{V}{\sqrt{gL}}$

Inverse of Reynolds number, where $\mathrm{Re} = \frac{\rho V L}{\mu}$

Thus, Eq. 10-5 can therefore be written as

Navier-Stokes Equation in Nondimensional Form:

$$[\mathrm{St}]\frac{\partial \vec{V}^*}{\partial t^*} + (\vec{V}^*\cdot\vec{\nabla}^*)\vec{V}^* = -[Eu]\vec{\nabla}^* P^* + \left[\frac{1}{\mathrm{Fr}^2}\right]\vec{g}^* + \left[\frac{1}{\mathrm{Re}}\right]\nabla^{*2}\vec{V}^* \qquad \textbf{(10–6)}$$

Nondimensionalization vs. ***Normalization***:

Equation 10-6 above is *nondimensional*, but not necessarily *normalized*. What is the difference?

- ***Nondimensionalization*** concerns only the ***dimensions*** of the equation – we can use *any* value of scaling parameters L, V, etc., and we always end up with Eq. 10-6.
- ***Normalization*** is more restrictive than nondimensionalization. To *normalize* the equation, we must choose scaling parameters L, V, etc. that are appropriate for the flow being analyzed, such that ***all nondimensional variables*** (t^*, $\vec{V}^*$, P^*, etc.) ***in Eq. 10-6 are of order of magnitude unity***. In other words, their minimum and maximum values are reasonably close to 1.0 (e.g., $-6 < P^* < 3$, or $0 < P^* < 11$, but *not* $0 < P^* < 0.001$, or $-200 < P^* < 500$). We express the normalization as follows:

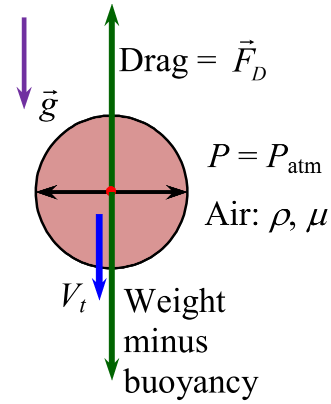
$$t^* \sim 1, \quad \vec{x}^* \sim 1, \quad \vec{V}^* \sim 1, \quad P^* \sim 1, \quad \vec{g}^* \sim 1, \quad \vec{\nabla}^* \sim 1$$

If we have properly normalized the Navier-Stokes equation, we can compare the *relative importance* of various terms in the equation by comparing the *relative magnitudes* of the nondimensional parameters St, Eu, Fr, and Re.

C. The Creeping Flow Approximation

Example: Terminal velocity of a settling air pollution particle

Given: An air pollution particle of diameter 40 microns (40×10^{-6} m) falls towards the ground. After a little while, it reaches **terminal settling velocity** V_t, which is its steady settling velocity in which aerodynamic drag force is balanced by its weight (minus buoyancy). The particle density is 1500 kg/m^3, the air density is 0.840 kg/m^3, and the air viscosity is 1.45×10^{-5} kg/(m s).



To do: Calculate V_t in m/s.

Solution:

We assume creeping flow, and then will need to check afterwards if the Reynolds number is small enough or not.

D. Approximation for Inviscid Regions of Flow

1. Definition of Inviscid Regions of Flow and the Euler Equation

Definition: An **inviscid region of flow** is a region of flow in which net viscous forces are negligible compared to pressure and/or inertial forces.

Let's look at our nondimensionalized Navier-Stokes equation for this case:

$$[\mathrm{St}]\frac{\partial \vec{V}*}{\partial t*}+\left(\vec{V}*\cdot\vec{\nabla}*\right)\vec{V}*=-[\mathrm{Eu}]\vec{\nabla}*P*+\left[\frac{1}{\mathrm{Fr}^2}\right]\vec{g}*+\left[\frac{1}{\mathrm{Re}}\right]\vec{\nabla}*^2\vec{V}*$$