Today, we will:

- Continue discussing inviscid regions of flow; the beloved Bernoulli equation (again)
- Discuss irrotational regions of flow

D. Approximation for Inviscid Regions of Flow (continued)

1. Definition of Inviscid Regions of Flow and the Euler Equation

Definition: An **inviscid region of flow** is a region of flow in which net viscous forces are negligible compared to pressure and/or inertial forces.

We obtained the Euler Equation,

$$\boxed{\rho\frac{D\vec{V}}{Dt} = \rho\left[\frac{\partial\vec{V}}{\partial t} + \left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}\right] = -\vec{\nabla}P + \rho\vec{g} \qquad (10\text{-}13)}$$

2. The "Beloved" Bernoulli equation in Inviscid Regions of Flow

Recall from Chapter 5, we derived the **Bernoulli equation** as a ***degenerate form of the energy equation*** for cases in which friction and other irreversible losses are negligible,

$$\boxed{\frac{P}{\rho} + \frac{V^2}{2} + gz = \text{constant along a streamline}}$$

It turns out that we can get this *same* equation by working on the Euler equation, and using some vector identities. [See text for derivation, some of which is shown here.]

Vector identity:

$$(\vec{V}\cdot\vec{\nabla})\vec{V} = \vec{\nabla}\left(\frac{V^2}{2}\right) - \vec{V}\times(\vec{\nabla}\times\vec{V}) \qquad \textbf{(10–14)}$$

Recognizing the vorticity vector, Eq. 10-13, the Euler equation, becomes

$$\vec{\nabla}\left(\frac{V^2}{2}\right) - \vec{V}\times\vec{\zeta} = -\frac{\vec{\nabla}P}{\rho} + \vec{g} = \vec{\nabla}\left(-\frac{P}{\rho}\right) + \vec{g} \qquad \textbf{(10–15)}$$

where we have divided each term by the density and moved ρ within the gradient operator, since density is constant in an incompressible flow.

We make the further assumption that gravity acts only in the $-z$-direction (Fig. 10–18), so that

$$\vec{g} = -g\vec{k} = -g\vec{\nabla}z = \vec{\nabla}(-gz) \qquad \textbf{(10–16)}$$

where we have used the fact that the gradient of coordinate z is unit vector $\vec{k}$ in the z-direction. Note also that g is a constant, which allows us to move it (and the negative sign) within the gradient operator. We substitute Eq. 10–16 into Eq. 10–15, and rearrange by combining three terms within one gradient operator,

$$\vec{\nabla}\left(\frac{P}{\rho} + \frac{V^2}{2} + gz\right) = \vec{V}\times\vec{\zeta} \qquad \textbf{(10–17)}$$

Summary, equations for 2-D, steady, incompressible, irrotational flow in the *x*-*y* plane:

$\boxed{\vec{\zeta} = \vec{\nabla} \times \vec{V} = 0} \rightarrow \boxed{\vec{V} = \vec{\nabla}\phi} \rightarrow \boxed{\nabla^2 \phi = 0}$; $\boxed{\nabla^2 \psi = 0}$ & $\boxed{\dfrac{P}{\rho} + \dfrac{V^2}{2} + gz = \text{constant everywhere}}$

Cartesian: $\boxed{\nabla^2 \phi = \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0}$, Cylindrical: $\boxed{\nabla^2 \phi = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r \dfrac{\partial \phi}{\partial r} \right) + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2} = 0}$

Cartesian: $\boxed{u = \dfrac{\partial \psi}{\partial y}}$ $\boxed{v = -\dfrac{\partial \psi}{\partial x}}$, Cylindrical planar ($r$-$\theta$ plane): $\boxed{u_r = \dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta}}$ $\boxed{u_\theta = -\dfrac{\partial \psi}{\partial r}}$