Today, we will:

- Discuss hydrostatic forces on submerged surfaces
- Do some example problems – hydrostatic forces on submerged surfaces

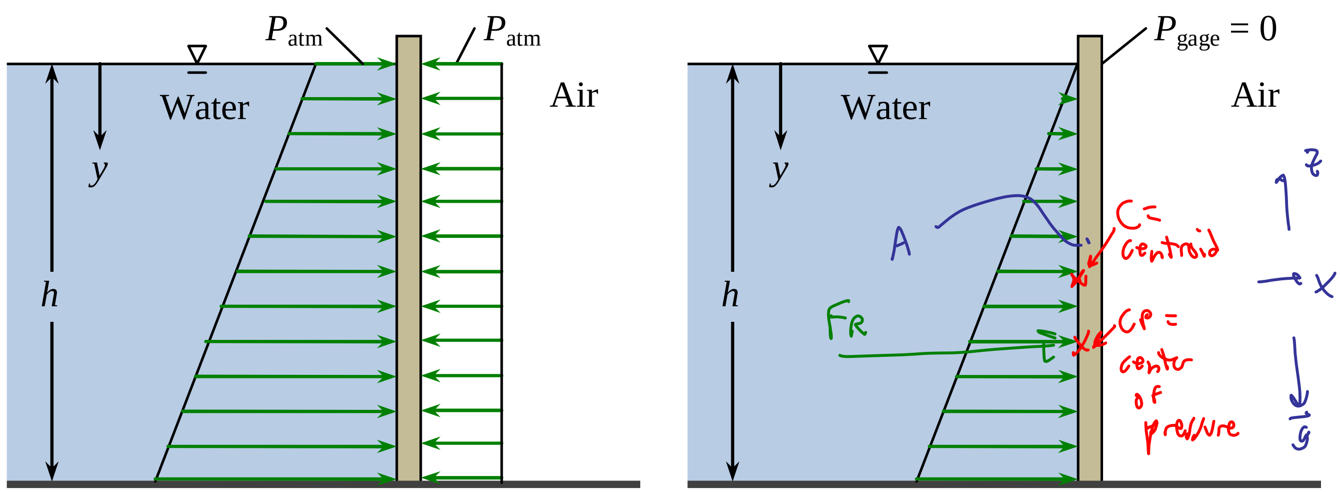
E. Hydrostatic Forces on Submerged Surfaces

1. Plane (flat) surfaces

Recall our fluid statics equation: $\boxed{P_{\text{below}} = P_{\text{above}} + \rho g |\Delta z|}$. → P increases linearly with depth

Example – the vertical wall of a rectangular container with a liquid in it.

$P_{gage} = P - P_{atm}$



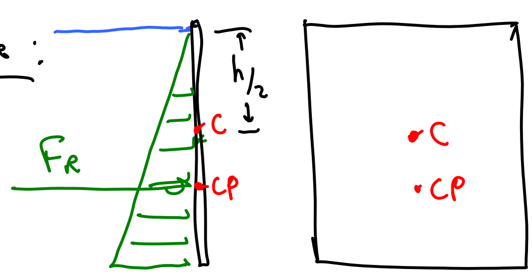
Absolute pressure distribution | Gage pressure distribution

Resultant pressure force = $\boxed{F_R = \int_A P_{gage}\, dA}$ in x direction

Let CP = center of pressure = location where F_R acts

Mag. of F_R is calculated at the centroid, C, but its location is at CP

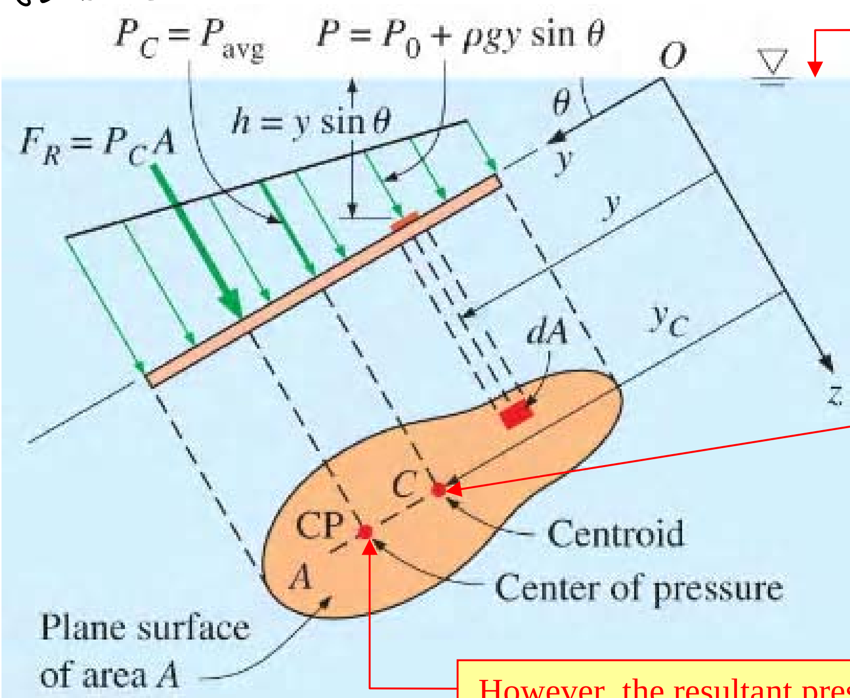
Rectangular plate:



$F_R = P_{gage,\,average} \cdot A$

Calculate at centroid, C

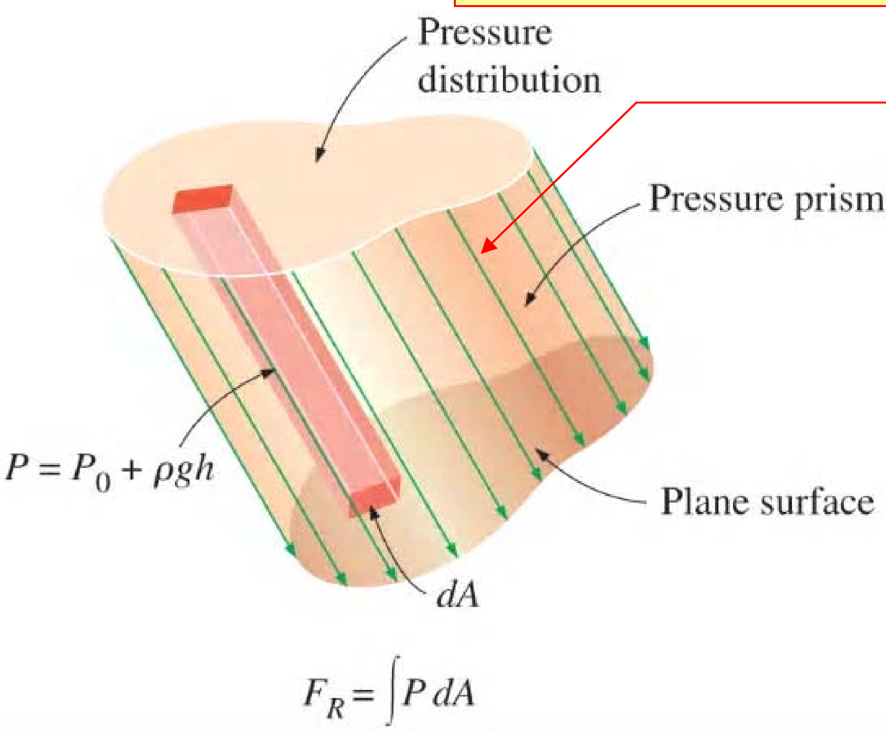
GENERAL CASE:



Liquid surface, open to air pressure P_0 (P_0 is usually P_{atm})

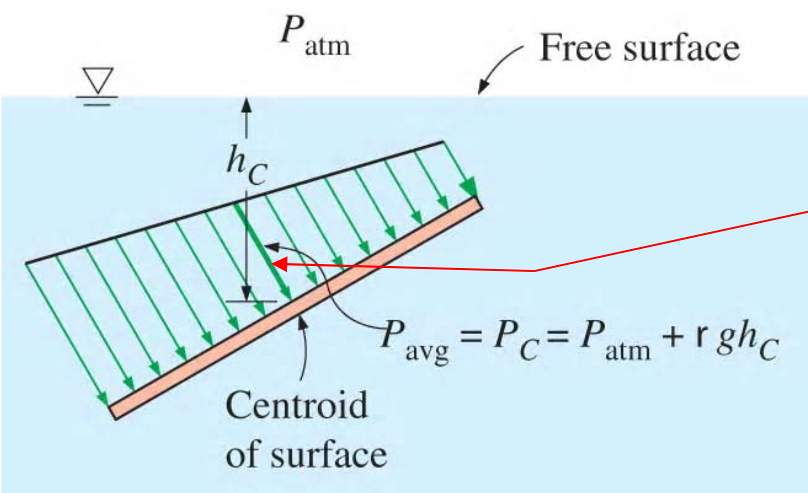
The **centroid** C is the mathematical center of the plate's area.
We calculate the average pressure at the centroid.

However, the resultant pressure force acts not at C, but at CP, the **center of pressure**.



Because pressure increases as you go down in the liquid, the "pressure prism" is thicker towards the bottom, and therefore, the center of pressure CP is located *below* the centroid C.

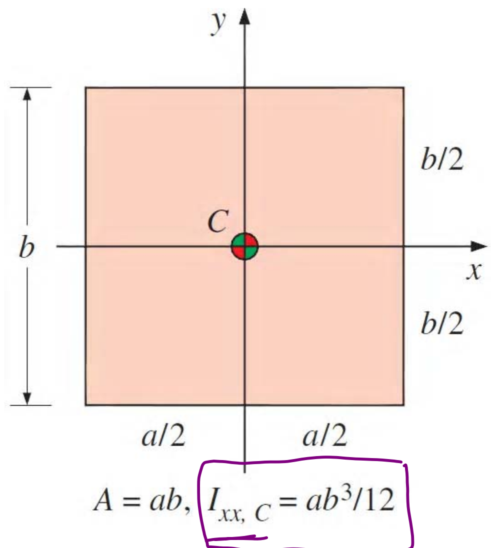
- Magnitude of $\vec{F}_R$ is calculated at the centroid $\boxed{F_R = P_C \cdot A}$
- Direction of $\vec{F}_R$ is normal to the plate
- Location of $\vec{F}_R$ is at the center of pressure, CP



Average pressure P_{ave} is easy to find – it is simply the pressure at the centroid C of the plate's surface area.

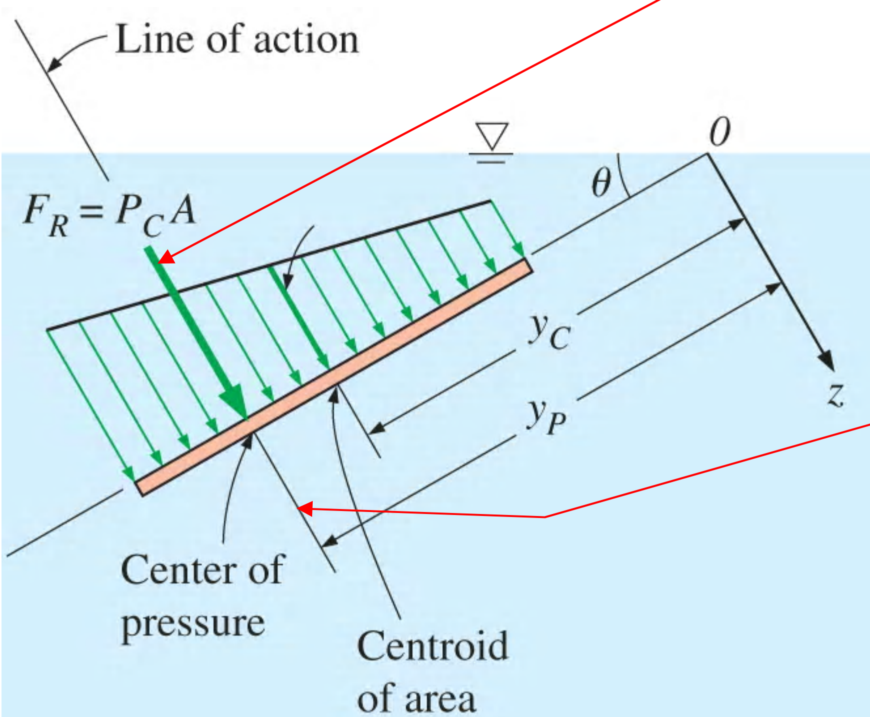
See the text, **Fig. 3-31**, for centroids and centroidal moments of inertia for some common shapes.

Example, for a rectangle, C is in the middle and $I_{xx,C} = ab^3/12$



From E. Mech. class

☆ $I_{xx,c}$ = centroidal moment (of inertia) which depends on geometry of the plate



The magnitude of resultant force F_R on the face of the plate is equal to the pressure P_C at the centroid C times the area of the plate,

$$F_R = P_C A$$

But, F_R does *not* act at the centroid! It acts at the center of pressure!

The line of action of the resultant hydrostatic force passes through the center of pressure CP and acts perpendicular to the plate.

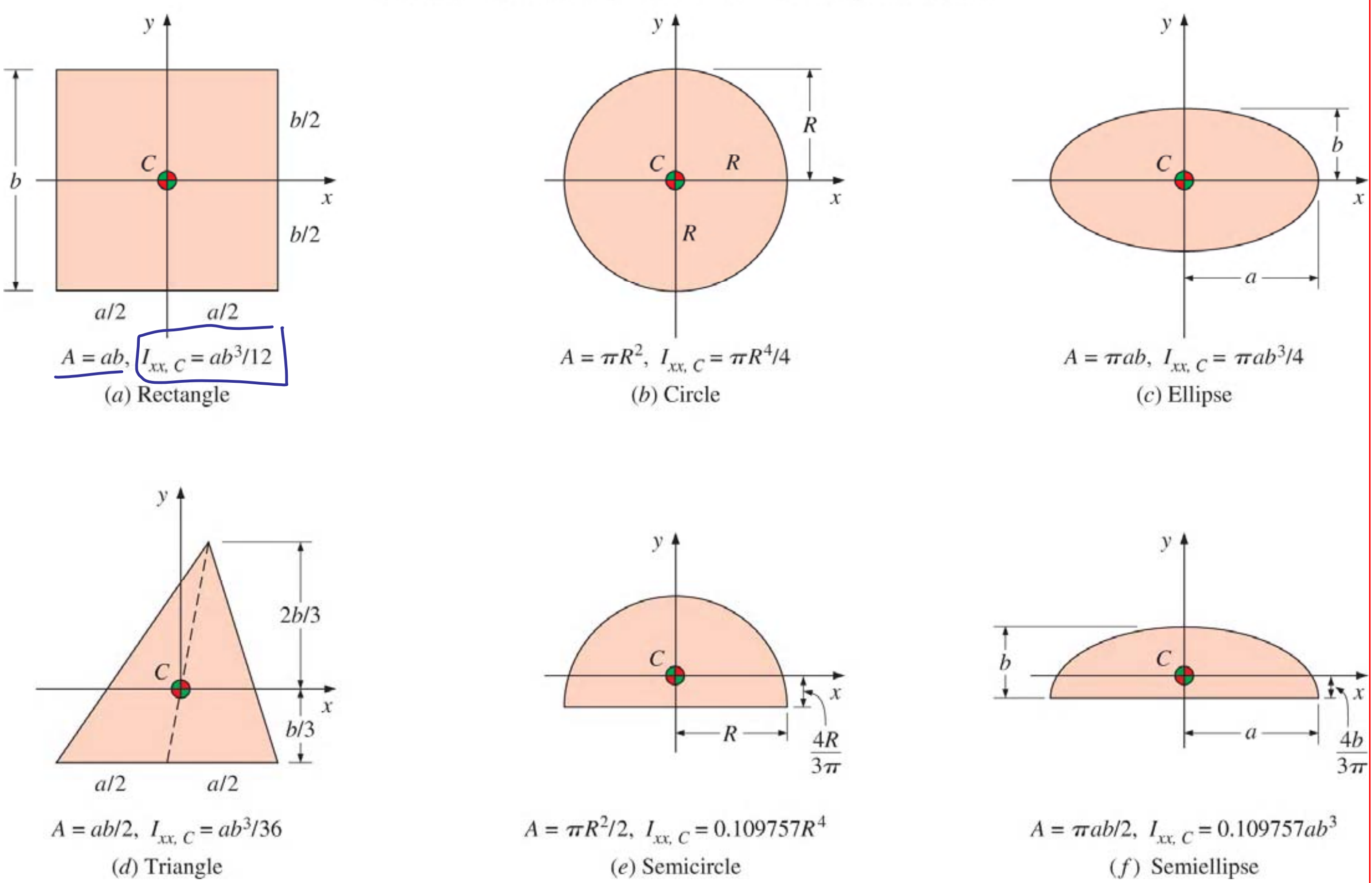


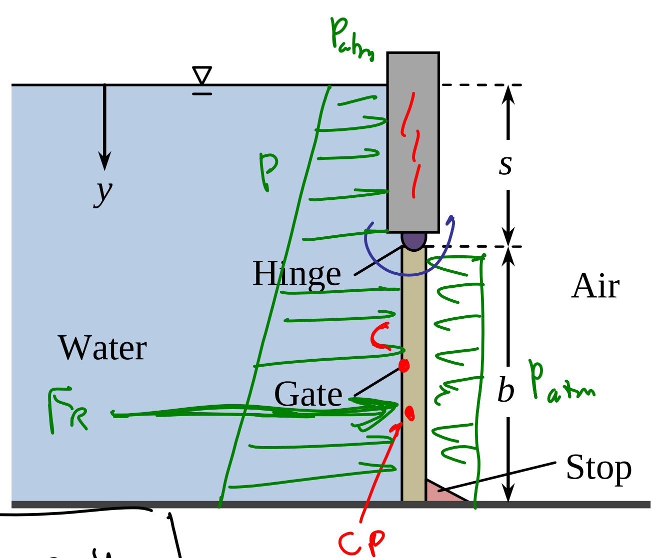
Figure 3-31. Centroids and centroidal moments of inertia for some common geometries.

Example: Force on a submerged gate

Given: A rectangular gate of height *b* and width *a* (into the page) holds back water in a reservoir. (The gate can swing open to let some water out when necessary.) The height from the water surface to the hinge is *s*.

To do: Calculate the resultant force on the gate and its location.

Solution:



Centroid C is at center of the plate

$$\boxed{y_c = s + \frac{b}{2}}$$

@ C, $\boxed{P_c = P_{atm} + \rho g y_c}$

$$\boxed{P_c = P_{atm} + \rho g (s + b/2)}$$

$F_R = P_c \cdot A = P_c \cdot a \cdot b$

on left side of plate

$F = P_{atm} \cdot a \cdot b$ on right side of plate

Net F_R due to pressure = $F_{left} - F_{right}$ $\boxed{F_R = \rho g(\delta + b/2)ab}$

Magnitude

Where is F_R acting? Eq (3.22b)

$$\boxed{y_p = y_c + \frac{I_{xx,c}}{y_c A}}$$

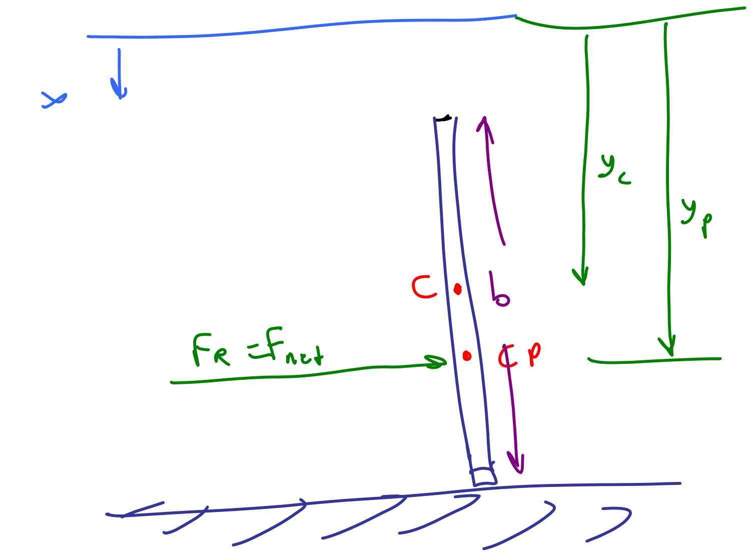
$I_{xx,c}$ = centroidal moment (of inertia)

here, $I_{xx,c} = \frac{ab^3}{12}$

@ CP

$$\boxed{y_p = \delta + \frac{b}{2} + \frac{b^2}{12(\delta + b/2)}}$$

location of net pressure force

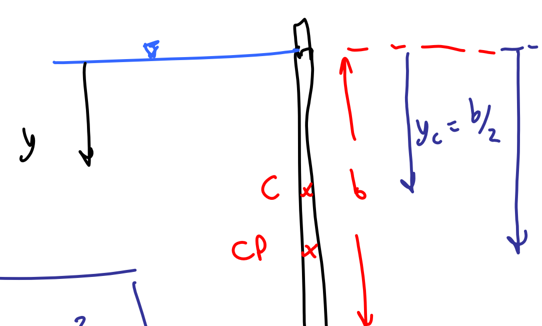


Special case, when plate is at surface

$$\boxed{F_{net} = \rho g \frac{b}{2}(ab)}$$

$$\boxed{y_c = b/2}$$

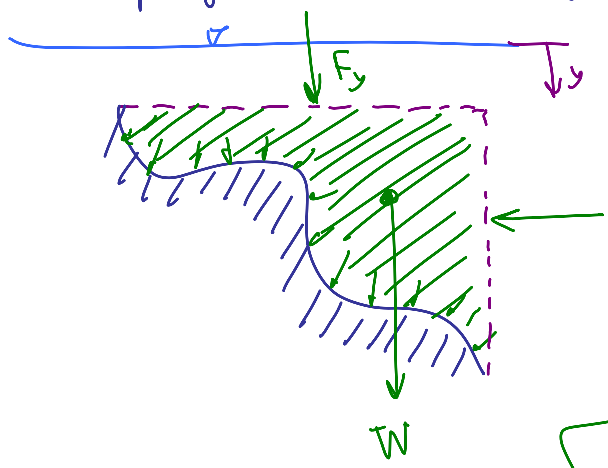
$$\boxed{y_{cp} = y_p = \frac{2}{3}b}$$

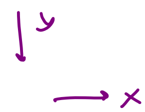


E. Hydrostatic Forces on Submerged Surfaces (continued)

1. Plane (flat) surfaces
2. Curved surfaces — more complex

☆ Pressure force on a curved surface is the same as that on its projected flat surface





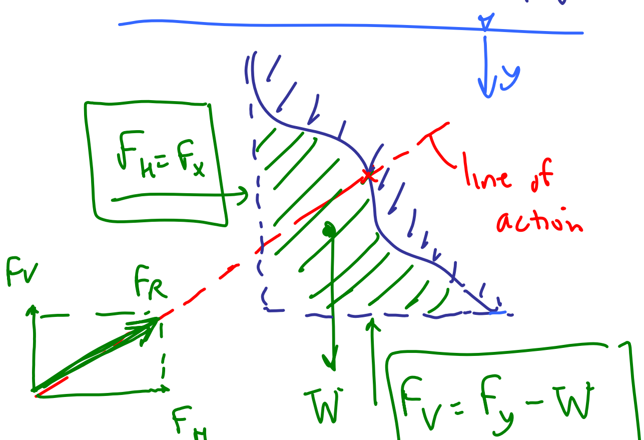
F_x = horizontal pressure force = same as pressure force on the vertical projection

$\boxed{F_H = F_x}$

F_V = vertical force on surface = force on projected horizontal area + Weight inside the shaded volume $\boxed{F_V = F_y + W}$

if surface is below the horizontal plane

☆ if surface is above the projected area, we subtract the weight



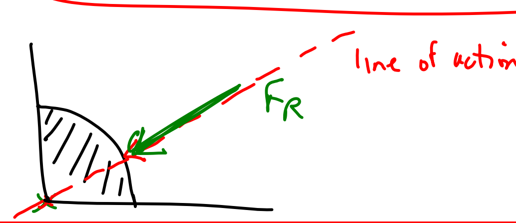
$\boxed{F_H = F_x}$

$\boxed{F_V = F_y - W}$

F_V & F_H can be calculated, & their vector sum is F_R, but we still need to know where it acts.

Calculate a line of action

For cylinders, the line of action is always through the center of the circle ☆

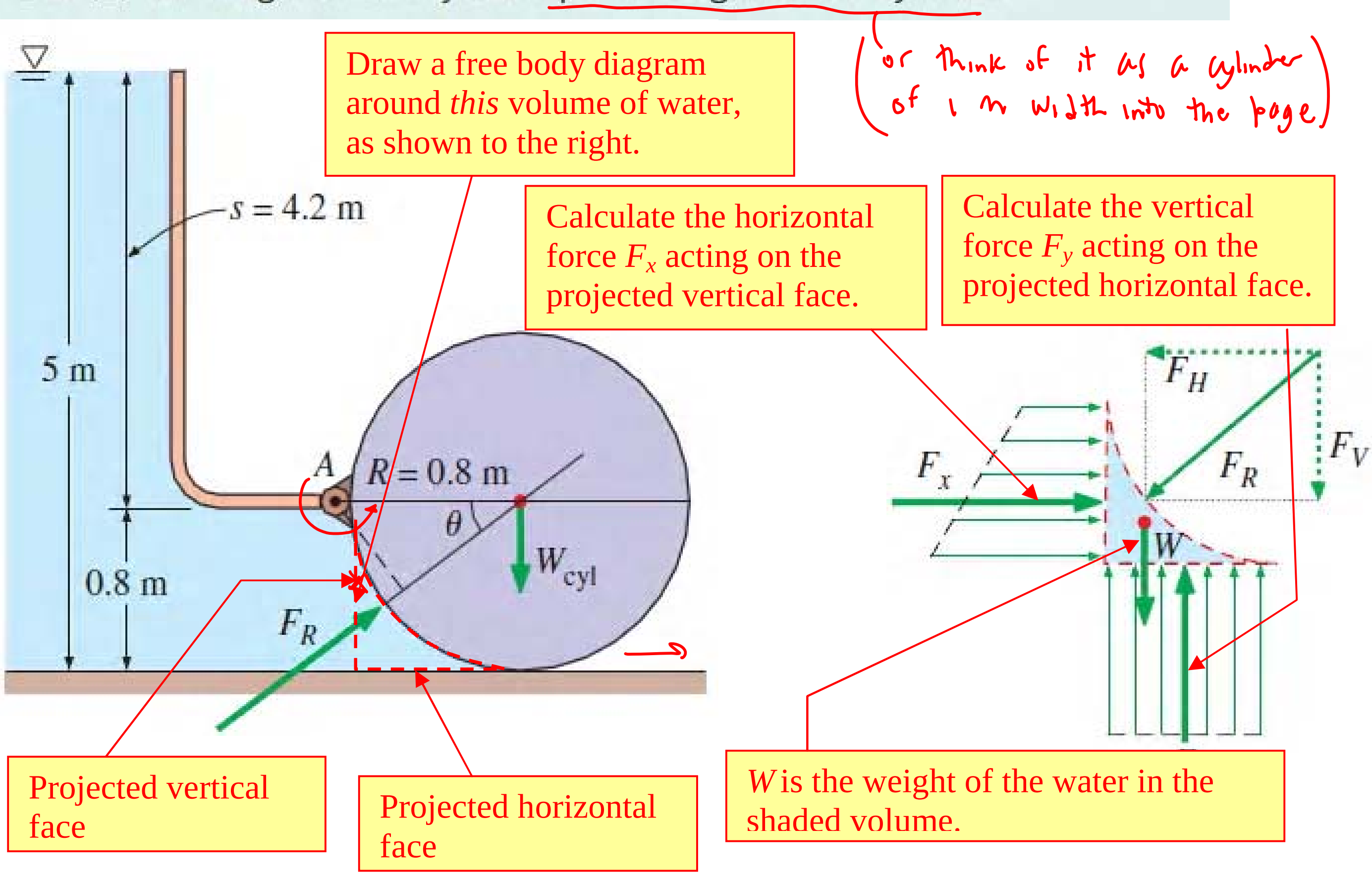


Example Problem – Hydrostatic Pressure Force on Curved Surfaces

(Example Problem 3-9, Çengel and Cimbala)

EXAMPLE 3–9 A Gravity-Controlled Cylindrical Gate

A long solid cylinder of radius 0.8 m hinged at point A is used as an automatic gate, as shown in Fig. 3–40. When the water level reaches 5 m, the gate opens by turning about the hinge at point A. Determine (*a*) the hydrostatic force acting on the cylinder and its line of action when the gate opens and (*b*) the weight of the cylinder per m length of the cylinder.



SOLUTION The height of a water reservoir is controlled by a cylindrical gate hinged to the reservoir. The hydrostatic force on the cylinder and the weight of the cylinder per m length are to be determined.

Assumptions **1** Friction at the hinge is negligible. **2** Atmospheric pressure acts on both sides of the gate, and thus it cancels out.

Properties We take the density of water to be 1000 kg/m^3 throughout.

Analysis (*a*) We consider the free-body diagram of the liquid block enclosed by the circular surface of the cylinder and its vertical and horizontal projections. The hydrostatic forces acting on the vertical and horizontal plane surfaces as well as the weight of the liquid block are determined as

Horizontal force on vertical surface:

> The centroid of the projected vertical surface is located at a depth of $s + R/2$ from the water surface.

$$F_H = F_x = P_{avg}A = \rho g h_C A = \rho g\underline{(s + R/2)}A$$

$$= (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(4.2 + 0.8/2\ \text{m})(0.8\ \text{m} \times 1\ \text{m})\left(\frac{1\ \text{kN}}{1000\ \text{kg}\cdot\text{m/s}^2}\right)$$

$$= 36.1\ \text{kN}$$

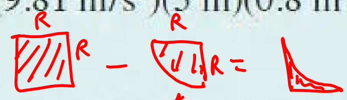
Vertical force on horizontal surface (upward):

> The centroid of the projected horizontal surface is located at a depth of h_{bottom} from the water surface. (P is constant along this surface since it is at a constant depth.)

$$F_y = P_{avg}A = \rho g h_C A = \rho g h_{\text{bottom}} A$$

$$= (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(5\ \text{m})(0.8\ \text{m} \times 1\ \text{m})\left(\frac{1\ \text{kN}}{1000\ \text{kg}\cdot\text{m/s}^2}\right)$$

$$= 39.2\ \text{kN}$$



[This is the area we want]

Weight (downward) of fluid block for one m width into the page:

$$W = mg = \rho g V = \rho g(\underbrace{R^2 - \pi R^2/4}_{\text{area of the chunk of water}})(1\ \text{m})$$

1 m width into the page

$$= (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(0.8\ \text{m})^2(1 - \pi/4)(1\ \text{m})\left(\frac{1\ \text{kN}}{1000\ \text{kg}\cdot\text{m/s}^2}\right)$$

$$= 1.3\ \text{kN}$$

> We subtract weight W since the shaded volume is *below* the curved surface.

Therefore, the net upward vertical force is

$$F_V = F_y - W = 39.2 - 1.3 = 37.9\ \text{kN}$$

Then the magnitude and direction of the hydrostatic force acting on the cylindrical surface become

$$F_R = \sqrt{F_H^2 + F_V^2} = \sqrt{36.1^2 + 37.9^2} = \mathbf{52.3\ kN}$$

$$\tan\theta = F_V/F_H = 37.9/36.1 = 1.05 \rightarrow \theta = 46.4°$$

Therefore, the magnitude of the hydrostatic force acting on the cylinder is 52.3 kN per m length of the cylinder, and its line of action passes through the center of the cylinder making an angle 46.4° with the horizontal.

> It turns out that for *cylindrical* surfaces (a circular arc shape), the resultant hydrostatic force acting on the surface always passes through the *center* of the circular arc.

Now we have the resultant force & its direction & location so we need to use moments to solve the problem.

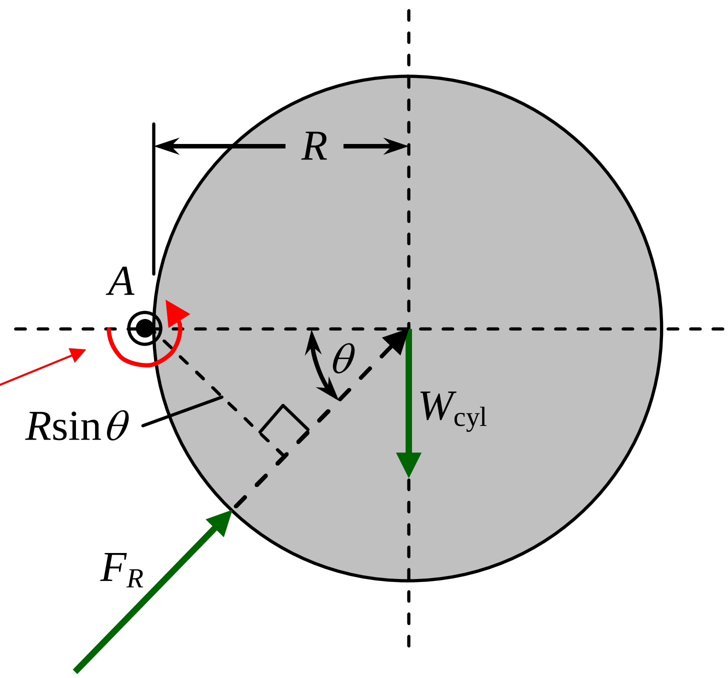
(*b*) When the water level is 5 m high, the gate is about to open and thus the reaction force at the bottom of the cylinder is zero. Then the forces other than those at the hinge acting on the cylinder are its weight, acting through the center, and the hydrostatic force exerted by water. Taking a moment about point *A* at the location of the hinge and equating it to zero gives

$$\underbrace{F_R R \sin\theta}_{(1)} - \underbrace{W_{cyl} R}_{(2)} = 0 \rightarrow W_{cyl} = F_R \sin\theta = (52.3 \text{ kN}) \sin 46.4° = \mathbf{37.9\ kN}$$

We take the moment about point *A*, using counterclockwise as positive. As shown in the sketch to the right, there are only two moments acting about point *A*:

(1) The net hydrostatic force acting on the portion of the cylinder that is in contact with the water times *its* moment arm. Its force is F_R and its moment arm is $R\sin\theta$ which is the perpendicular distance from *A* to the line of action of the force. (This moment is positive.)

(2) The weight of the cylinder times its moment arm, which is the radius of the cylinder. (This moment is negative.)



Discussion The weight of the cylinder per m length is determined to be 37.9 kN. It can be shown that this corresponds to a mass of 3863 kg per m length and to a density of 1921 kg/m³ for the material of the cylinder.