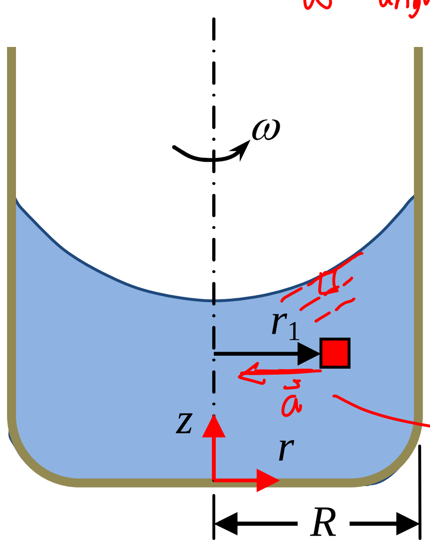
Today, we will:

- Discuss fluids in rigid-body *rotation*
- Begin Chapter 4 – FLUID KINEMATICS
- Discuss the material acceleration and the material derivative, and show examples
- Discuss various kinds of flow patterns and flow visualization techniques

3. Rigid-body rotation

Consider a container of liquid of radius *R* spinning at constant angular velocity ω (angular velocity vector is straight up as shown).

$\vec{\omega}$ = angular velocity vector ↑ up ↓ $\vec{g}$



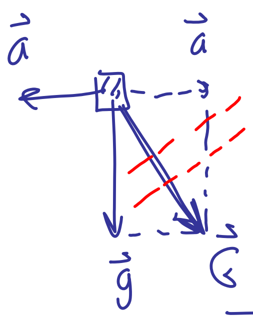
Eq. for rigid body motion:

$$\boxed{\vec{\nabla} P = \rho(\vec{g} - \vec{a}) = \rho \vec{G}}$$

$\vec{a}$ = centripetal accel.

$$\boxed{a = r\omega^2}$$

as r ↑ a ↑, so G tilts more

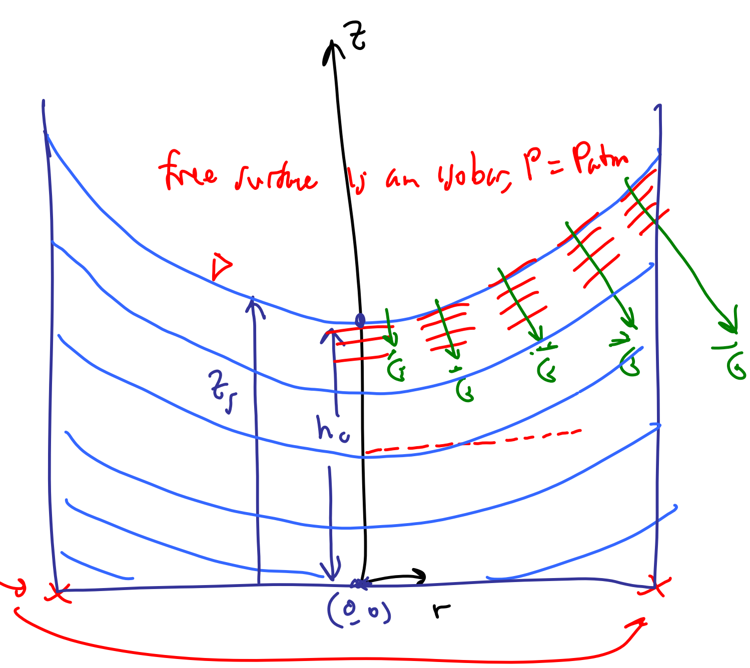


$\vec{G} = \vec{g} - \vec{a}$

Surface is a paraboloid
(3-D parabola)

($\vec{G}$ increases with r and tilts more with r, as sketched)

highest P



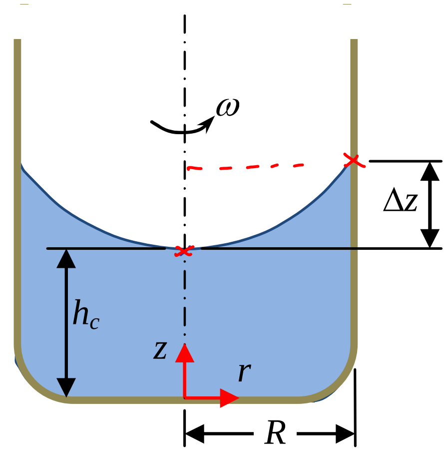
Free surface → $\boxed{z_s = \frac{\omega^2 r^2}{2g} + h_c}$

Isobars

$$\boxed{z_{isobar} = \frac{\omega^2 r^2}{2g} + const}$$

☆ always use $\frac{rad}{s}$ for ω

Example: Rigid-body rotation



Given: A container of water spins at rotation rate $\dot{n} = 100$ rpm. The radius of the container is R = 11.05 cm. The surface height at the center is h_c = 4.62 cm.

To do: Calculate the elevation distance Δz (in units of cm) between the water surface at the center of the paraboloid and at the rim of the paraboloid. (to 3 digits)

Solution:

First, we must convert rpm to radians/s:

$$\omega = \left(100 \frac{\text{rot}}{\text{min}}\right)\left(\frac{2\pi \text{ rad}}{\text{rot}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 10.47198 \frac{\text{rad}}{\text{s}}$$

Recall, the equation for the free surface of the liquid in rigid-body rotation,

$$\boxed{z_s = \frac{\omega^2 r^2}{2g} + h_c}$$

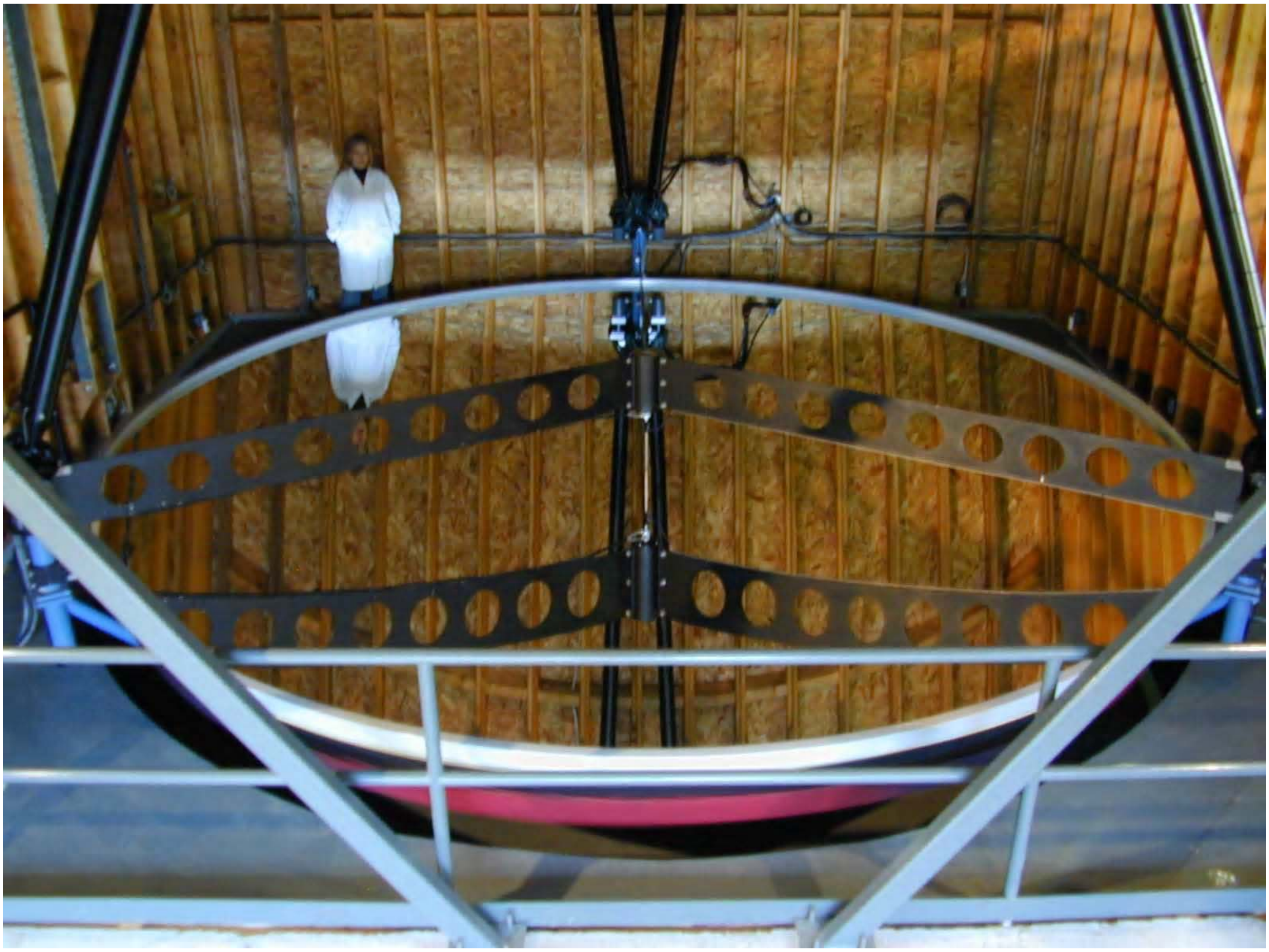
$$\Delta z = z_{s_{r=R}} - z_{s_{r=0}}$$

$$\Delta z = \boxed{\frac{\omega^2 R^2}{2g}} + \cancel{h_c} - \left(0 + \cancel{h_c}\right)$$

$$\Delta z = \frac{\left(10.47198 \frac{\text{rad}}{\text{s}}\right)^2 (11.05 \text{ cm})^2}{2\left(9.807 \text{ m/s}^2\right)}\left(\frac{1 \text{ m}}{100 \text{ cm}}\right) = \boxed{6.83 \text{ cm}}$$

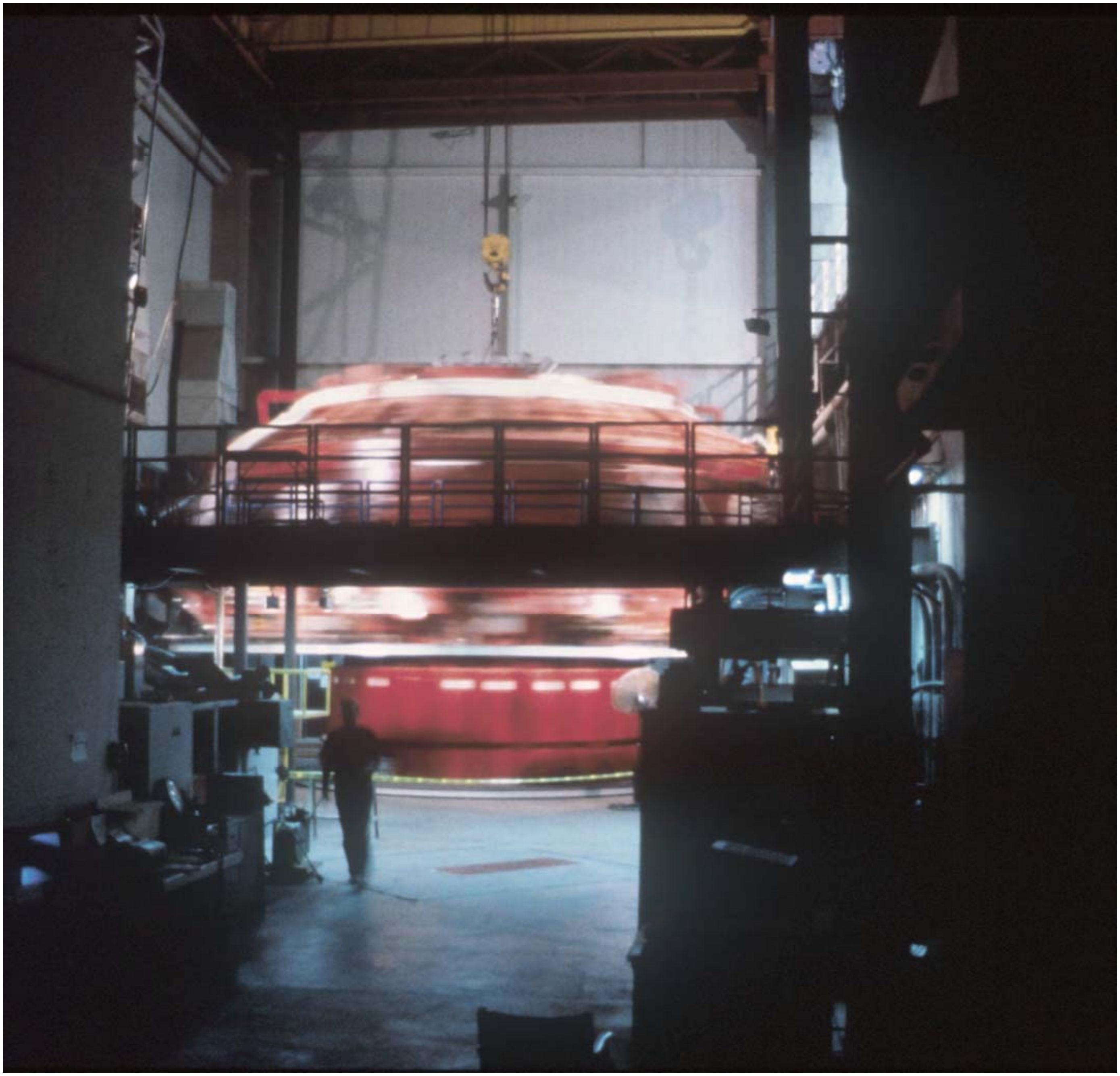
Liquid mercury mirrors. By rotating a container of mercury, a nice parabolic mirror can be generated without the need to grind or polish. Unfortunately, it can look only straight up. However, there is some discussion of creating similar mirrors in space – thrust can be used in place of gravity to produce the parabolic shape.

Example: The Large Zenith Telescope in Canada: Photo from http://www.astro.ubc.ca/LMT/lzt/index.html.



Mirrors made from spinning molten glass in a furnace, and letting it harden in the paraboloid shape.

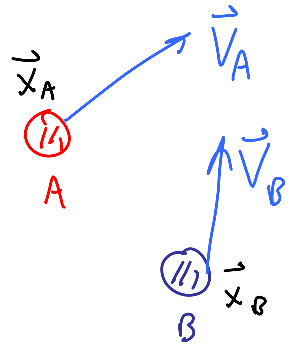
Example: The 40-foot (12 meter) diameter spinning furnace used in casting 6.5 meter and 8.4 meter borosilicate glass "honeycomb" mirrors at the Steward Observatory Mirror Lab, University of Arizona. Image from http://uanews.org.



III. FLUID KINEMATICS (Ch. 4)

A. Descriptions of Fluid Flow – there are two ways to describe fluid flow:

1. Lagrangian description → Follow & keep track of individual fluid particles as fn's of time



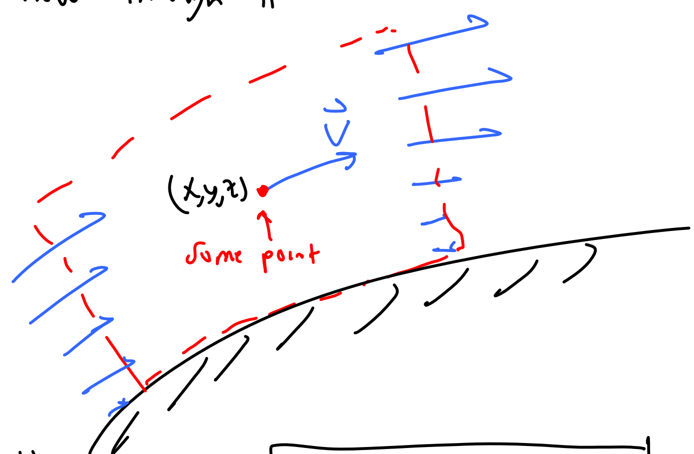
→ keep track of $\vec{x}_A$ & $\vec{x}_B$ etc.. vs time

In F.M., almost impossible to track enough particles

Lagrangian description is not often used in F.M.

2. Eulerian description → more common

We identify a region of the flow → control volume & watch fluid flow through it



We Describe Flow Field Variables → eg.

$$\boxed{\vec{V} = \vec{V}(x,y,z,t)} = \text{velocity field}$$

accel. vector → $\boxed{\vec{a} = \vec{a}(x,y,z,t)}$

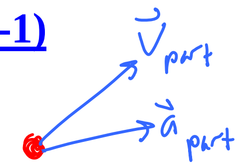
$$\boxed{P = P(x,y,z,t)} = \text{pressure field}$$

Eulerian description is usually preferred in fluid mechanics

3. Acceleration field and material derivative

Derivation of Material Acceleration (Section 4-1)

Goal → transform $\vec{a}_{particle}$ into a field variable $\vec{a}(x,y,z,t)$



Acceleration of a fluid particle:

$$\boxed{\vec{a}_{\text{particle}} = \frac{d\vec{V}_{\text{particle}}}{dt}} \tag{4–6}$$

This is a *Lagrangian* description of the acceleration of a fluid particle.

However, at any instant in time t, the velocity of the particle is the same as the local value of the velocity *field* at the location $(x_{\text{particle}}(t), y_{\text{particle}}(t), z_{\text{particle}}(t))$ of the particle, since the fluid particle moves with the fluid by definition. In other words, $\vec{V}_{\text{particle}}(t) \equiv \vec{V}(x_{\text{particle}}(t), y_{\text{particle}}(t), z_{\text{particle}}(t), t)$. To take the time derivative in Eq. 4–6, we must therefore use the *chain rule*, since the dependent variable ($\vec{V}$) is a function of *four* independent variables ($x_{\text{particle}}, y_{\text{particle}}, z_{\text{particle}}$, and t),

Recall the ***chain rule***: If f is a function of two variables, t and some variable s which is itself also a function of t, then we take the total derivative of f with respect to t as follows:

$$\frac{df}{dt} = \frac{\partial f}{\partial t}\frac{dt}{dt} + \frac{\partial f}{\partial s}\frac{ds}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial s}\frac{ds}{dt}$$

Now let's apply this chain rule to the time derivative of the fluid particle's velocity:

Note that from the Lagrangian description (following a fluid particle, x_{particle} is a function of time, since the particle's location changes with time. Thus, $x_{\text{particle}} = x_{\text{particle}}(t)$. Similarly, $y_{\text{particle}} = y_{\text{particle}}(t)$ and $z_{\text{particle}} = z_{\text{particle}}(t)$.

Thus, the acceleration of a fluid particle is calculated using the chain rule as follows:

$$\vec{a}_{\text{particle}} = \frac{d\vec{V}_{\text{particle}}}{dt} = \frac{d\vec{V}}{dt} = \frac{d\vec{V}(x_{\text{particle}}, y_{\text{particle}}, z_{\text{particle}}, t)}{dt}$$

$$= \frac{\partial \vec{V}}{\partial t}\frac{dt}{dt} + \frac{\partial \vec{V}}{\partial x_{\text{particle}}}\frac{dx_{\text{particle}}}{dt} + \frac{\partial \vec{V}}{\partial y_{\text{particle}}}\frac{dy_{\text{particle}}}{dt} + \frac{\partial \vec{V}}{\partial z_{\text{particle}}}\frac{dz_{\text{particle}}}{dt} \tag{4–7}$$

$dt/dt =$ 1

$dx_{\text{particle}}/dt =$ u

$dy_{\text{particle}}/dt =$ v

$dz_{\text{particle}}/dt =$ w

Or, finally,

$$\boxed{\vec{a}_{\text{particle}}(x, y, z, t) = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}} \tag{4–8}$$

At the instant in time being considered,

$$\boxed{\vec{a}(x,y,z,t) = \vec{a}_{particle}(x,y,z,t)} \quad \text{(Eq 4-8)}$$

accel field — accel of a fluid particle

The accel field is thus

$$\boxed{\vec{a}(x,y,z,t) = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{(\vec{V}\cdot\vec{\nabla})\vec{V}}}$$

gradient operator

Material acceleration & Material derivative

$$\vec{a} = \frac{d\vec{V}}{dt} \rightarrow \text{write as } \boxed{\vec{a} = \frac{D\vec{V}}{Dt}}$$

notation emphasizes that this is the acceleration following a fluid particle

$$\boxed{\vec{a} = \frac{D\vec{V}}{Dt} = \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{local or unsteady part}} + \underbrace{(\vec{V}\cdot\vec{\nabla})\vec{V}}_{\text{advective or convective part}}}$$

local or unsteady part (due to unsteadiness)

advective or convective part (due to movement to a different part of the flow)

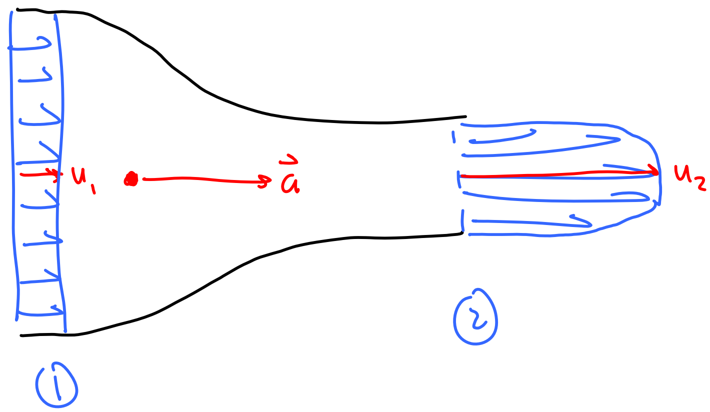
In general, $\frac{D}{Dt}$ is the material derivative & can operate on any variable

$$\boxed{\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{V}\cdot\vec{\nabla})}$$

Here we operate on $\vec{V}$

In a steady flow, $\frac{\partial}{\partial t}(\sim) = 0$, But advective part can still be non-zero

Physical example: Steady converging duct flow



particle is speeding up

$\therefore$ accelerating

$$\boxed{\therefore \vec{a} \neq 0}$$

even though flow is steady

Mathematical example:

Given 2-D flow (steady)

$$\boxed{\vec{V}(x,y) = 3x\,\vec{i} - 3y\,\vec{j}}$$

$$\begin{pmatrix} u = 3x \\ v = -3y \\ w = 0 \end{pmatrix}$$

To do: calc. steady accel field, $\vec{a}(x,y)$

Sol'n: Wrong → $\vec{a} = \dfrac{d\vec{V}}{dt} = 0$ since $\vec{V} \neq$ fnc. of time ✗

Correct → use material accel

$$\vec{a} = \frac{D\vec{V}}{Dt} = \cancel{\frac{\partial \vec{V}}{\partial t}} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

$$\underset{(\text{steady})}{0} \qquad (3x)(3\vec{i}) + (-3y)(-3\vec{j}) + 0$$

$$\boxed{\vec{a} = 9x\,\vec{i} + 9y\,\vec{j}}$$

☆ Accel field is not necessarily zero, even in a steady flow