Today, we will:
- Finish our example problem – rates of motion and deformation of fluid particles
- Discuss the Reynolds Transport Theorem (RTT)
- Show how the RTT applies to the conservation laws
- Begin Chapter 5 – Conservation Laws

Example: Rates of motion and deformation (**Continuation of previous example**)

Given: A two-dimensional velocity field in the *x-y* plane: $\vec{V} = (u,v) = 3x\vec{i} - 3y\vec{j}$ ($w = 0$).

To do: Calculate (**a**) rate of translation, (**b**) rate of rotation, (**c**) linear strain rate, (**d**) the shear strain rate, and (**e**) the strain rate tensor.

$u = 3x \qquad v = -3y$

Solution: We did Parts (**a**) and (**b**) already. Recall,

(**a**) The rate of translation is simply the velocity vector, $\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$. Here,

the rate of translation $= \boxed{\vec{V} = 3x\vec{i} - 3y\vec{j}}$.

(**b**) The rate of rotation is $\vec{\omega} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$. Here,

the rate of rotation is $\boxed{\vec{\omega} = 0}$, and the vorticity $= \vec{\zeta} = 2\vec{\omega} = 0$. This flow is *irrotational*.
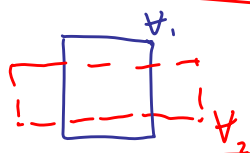
(**c**) The three components of linear strain rate are

$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z}$

$\boxed{\varepsilon_{xx} = 3 \quad \varepsilon_{yy} = -3 \quad \varepsilon_{zz} = 0}$

Notice that $\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$

$= 3 - 3 + 0 = \underline{\underline{0}}$

When the three linear strain rates add to zero, the flow is incompressible ✕

$\rightarrow \Psi_1 = \Psi_2$ (Volume remains constant)

Define: Volumetric strain rate $= \frac{1}{\Psi}\frac{D\Psi}{Dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$

$=$ rate of change of volume of a fluid particle per unit volume following a fluid particle

If $\frac{1}{\Psi}\frac{D\Psi}{Dt} = 0$, then the flow is incompressible

If " $\neq 0$, " " " compressible

$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z}$

$\varepsilon_{yy} = \frac{\partial v}{\partial y}$

Given: $\boxed{\vec{V} = 2xy\,\vec{i} - y^2\,\vec{j}}$ → Is it Ⓐ compressible

Ⓑ incompressible

$$\frac{1}{\cancel{V}}\frac{d\cancel{\rho V}}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$2y - 2y + 0 = \underline{\underline{0}} \longrightarrow \boxed{\text{Incompressible}}$$

(**d**) The three components of shear strain rate are

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \quad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$
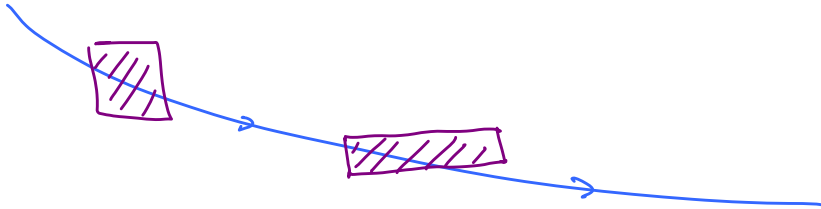
$$\underline{\begin{array}{l} u = 3x \\ v = -3y \\ w = 0 \end{array}}$$

2-(1)   2-0   2-0   2-0

$\varepsilon_{xy} = \frac{1}{2}\left(0 + 0\right)$ 0   0

$$\boxed{\varepsilon_{xy} = 0 \qquad \varepsilon_{zx} = 0 \quad \varepsilon_{yz} = 0}$$

$\underline{\underline{No\ shear\ strain}}$

(**e**) The strain rate tensor
We can conveniently combine the linear strain rates and the shear strain rates into one 9-component matrix, which is actually a 3x3 tensor called the ***strain rate tensor***:

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial x}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\right) & \dfrac{\partial v}{\partial y} & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial y}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}+\dfrac{\partial u}{\partial z}\right) & \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}+\dfrac{\partial v}{\partial z}\right) & \dfrac{\partial w}{\partial z} \end{pmatrix}$$

$$\underline{\begin{array}{l} u = 3x \\ v = -3y \\ w = 0 \end{array}}$$

Shear Strain rates

linar Strain rates

$$\boxed{\varepsilon_{ij} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

These are the $\underline{principle\ strain\ rates}$

$\underline{\underline{X\text{-}y\ are\ principle\ axes}}$

D. **The Reynolds Transport Theorem** (RTT) (Section 4-6)
   1. Introduction and derivation

The Reynolds Transport Theorem (RTT) (Section 4-6)

Recall from Thermodynamics:
- A *system* [also called a *closed system*] is a quantity of matter of fixed identity. *No mass can cross a system boundary.*
- A *control volume* [also called an *open system*] is a region in space chosen for study. *Mass **can** cross a control surface* (the surface of the control volume).
- The fundamental conservation laws (conservation of mass, energy, and momentum) *apply directly to systems*.
- However, in most fluid mechanics problems, *control volume analysis is preferred over system analysis* (for the same reason that the Eulerian description is usually preferred over the Lagrangian description).
- Therefore, we need to *transform the conservation laws from a system to a control volume*. This is accomplished with the *Reynolds transport theorem (RTT)*.

There is a direct **analogy** between the transformation from Lagrangian to Eulerian descriptions (for differential analysis using infinitesimally small fluid elements) and the transformation from systems to control volumes (for integral analysis using large, finite flow fields):
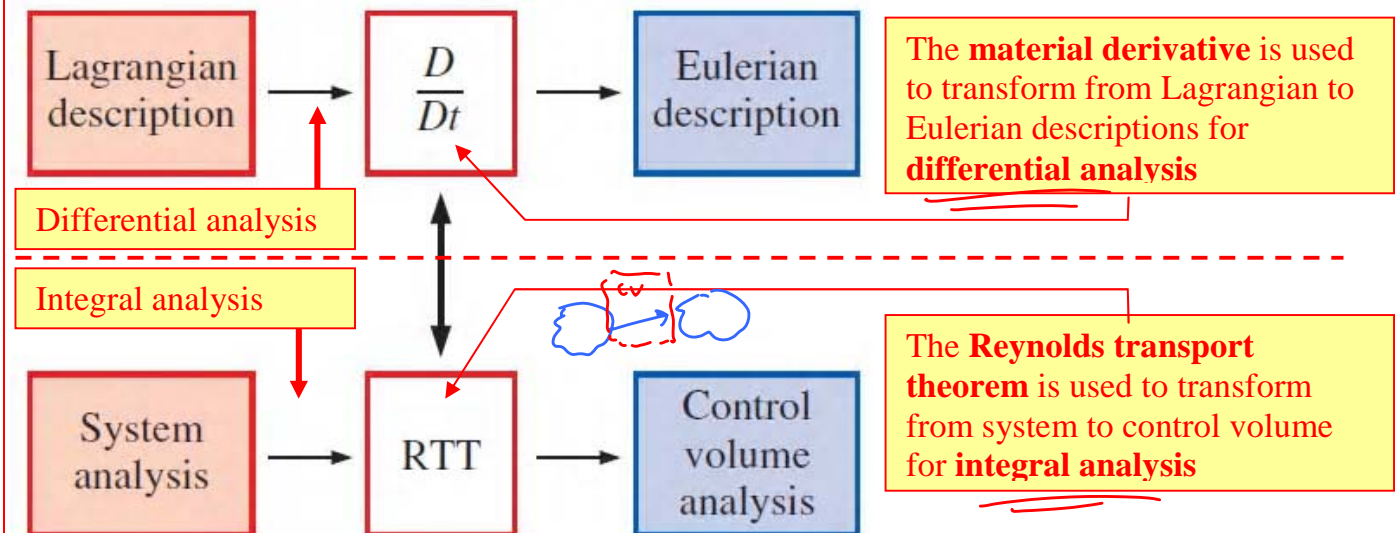

In both cases, the fundamental laws of physics (conservation laws) are known in the analysis on the left (Lagrangian or system), and must be transformed so as to be useful in the analysis on the right (Eulerian or control volume).

Another way to think about the RTT is that it is a *link* between the system approach and the control volume approach:

See text for detailed derivation of the RTT. Some highlights:

- Let B represent any extensive property (like mass, energy, or momentum).
- Let b be the corresponding intensive property, i.e., $b = B/m$ (property B per unit mass).
- Our goal is to find a relationship between B_{sys} or b_{sys} (property of the system, for which we know the conservation laws) and B_{CV} or b_{CV} (property of the control volume, which we prefer to use in our analysis).
- The results are shown below in various forms:
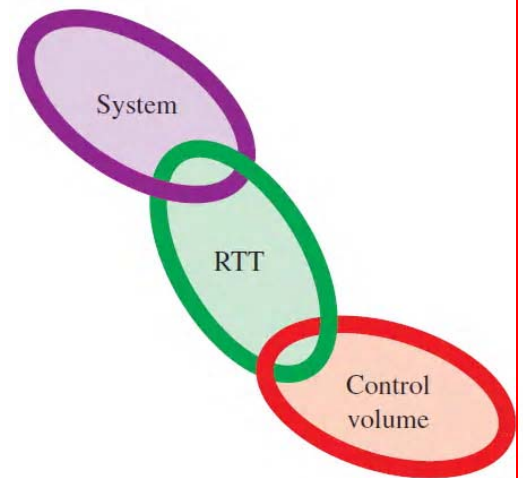
For **fixed** (non-moving and non-deforming) control volumes,

RTT, fixed CV: $$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b\, dV + \int_{CS} \rho b\, \vec{V}\cdot\vec{n}\, dA \qquad (4\text{--}41)$$

Alternate RTT, fixed CV: $$\frac{dB_{sys}}{dt} = \int_{CV} \frac{\partial}{\partial t}(\rho b)\, dV + \int_{CS} \rho b\, \vec{V}\cdot\vec{n}\, dA \qquad (4\text{--}42)$$

Since the control volume is *fixed*, the order of integration or differentiation does not matter, i.e., $\frac{d}{dt}\int_{CV} ...$ is the same as $\int_{CV} \frac{\partial}{\partial t}...$. Thus, the two circled quantities above are *equivalent* for a fixed control volume.

For **nonfixed** (moving and/or deforming) control volumes,

RTT, nonfixed CV: $$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b\, dV + \int_{CS} \rho b\, \vec{V_r}\cdot\vec{n}\, dA \qquad (4\text{--}44)$$

Note: The only difference in the equations is that we replace $\vec{V}$ by $\vec{V_r}$ in this version of the RTT for a moving and/or deforming control volume.

where $\vec{V_r}$ is the **relative velocity,** i.e., the velocity of the fluid *relative to the control surface* (which may be moving or deforming),

Relative velocity: $$\vec{V_r} = \vec{V} - \vec{V}_{CS} \qquad (4\text{--}43)$$

We can also switch the order of the time derivative and the integral in the first term on the right, but only if we use the *absolute* (rather than the relative) velocity in the second term on the right, i.e.,

Alternate RTT, nonfixed CV: $\qquad \dfrac{dB_{sys}}{dt} = \displaystyle\int_{CV} \dfrac{\partial}{\partial t}(\rho b)\, dV + \int_{CS} \rho b \vec{V}\cdot\vec{n}\, dA \qquad$ (4–45)

Comparing Eqs. 4-45 and 4-42, we see that they are identical. Thus, the most general form of the RTT that *applies to both fixed and non-fixed control volumes* is

General RTT, nonfixed CV: $\qquad \dfrac{dB_{sys}}{dt} = \displaystyle\int_{CV} \dfrac{\partial}{\partial t}(\rho b)\, dV + \int_{CS} \rho b \vec{V}\cdot\vec{n}\, dA \qquad$ (4–53)

Even though this equation is most general, it is often easier *in practice* to use Eq. 4-44 for moving and/or deforming (non-fixed) control volumes because the algebra is easier.

Simplifications: ☆ They cover — mostly we these
- For **steady** flow, the volume integral drops out. In terms of relative velocity,

RTT, steady flow: $\qquad \boxed{\dfrac{dB_{sys}}{dt} = \displaystyle\int_{CS} \rho b \vec{V_r}\cdot\vec{n}\, dA} \qquad$ (4–46)

- For control volumes where there are **well-defined inlets and outlets**, the control surface integral can be simplified, avoiding cumbersome integrations,

  Approximate RTT for well-defined inlets and outlets:

  Velocity

  $$\dfrac{dB_{sys}}{dt} = \dfrac{d}{dt}\int_{CV} \rho b\, dV + \sum_{out} \underbrace{\rho_{avg}b_{avg}V_{r,avg}A}_{\text{for each outlet}} - \sum_{in} \underbrace{\rho_{avg}b_{avg}V_{r,avg}A}_{\text{for each inlet}} \qquad (4\text{–}48)$$

  Volume

Note that the above equation is *approximate*, and may not always be accurate, but will be used almost exclusively in this course, and is used generally in engineering analysis.

$\forall$ = volume

V = velocity

$\dot{W}_{work}$

OUT

IN   CV

---
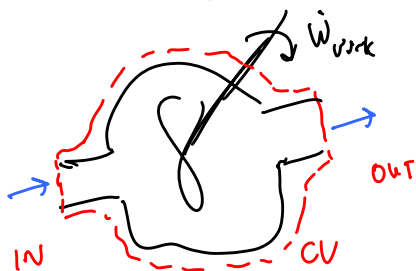
2. APPLICATIONS OF THE RTT

   a. Cons. of mass   For a System $\boxed{\dfrac{dm_{sys}}{dt} = 0}$ ☆

   Mass of a sys. is constant

APPLY RTT: (for a fixed CV)

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b \, d\forall + \int_{CS} \rho b \left(\vec{V}\cdot\vec{n}\right) dA$$

Let $\beta = m = mass$

$b = \beta/m = m/m = 1$

$\therefore \quad \frac{dm_{sys}}{dt} = 0 = \frac{d}{dt}\int_{CV} \rho \, d\forall + \int_{CS} \rho \left(\vec{V}\cdot\vec{n}\right) dA$

System eq.

transformed into a CV eq.

☆ This is our CV. eq. for cons. of mass for a fixed CV.

---

b. Cons. of energy   System → 1st law of thermo

$$\frac{dE_{sys}}{dt} = \dot{Q}_{net\ in} + \dot{W}_{net\ in}$$ For a System

specific total energy

Apply RTT — let $\beta = E$ $b = E/m = e$

total energy

☆ This is our energy eq. for a fixed CV

For a fixed CV, RTT gives

$$\frac{dE_{sys}}{dt} = \dot{Q}_{net\ in} + \dot{W}_{net\ in} = \frac{d}{dt}\int_{CV} \rho e \, d\forall + \int_{CS} \rho e \left(\vec{V}\cdot\vec{n}\right) dA \quad ☆$$

System eq.

CV. eq. for cons. of energy (fixed CV)

c. <u>Cons. of Linear momentum</u>

For a System → Newton's 2nd law

$$\frac{d}{dt}\left(m\vec{V}\right)_{sys} = \sum\vec{F}$$

$$m_{sys}\frac{d\vec{V}_{sys}}{dt} + \vec{V}_{sys}\cancel{\frac{dm_{sys}}{dt}}^{0} \implies \sum\vec{F}=m\vec{a}$$

$\vec{a}_{sys}$

Use RTT → Set $\quad B = m\vec{V} = $ linear momentum

$$b = B/m = \vec{V} = \text{velocity}$$

RTT becomes:

$$\frac{d}{dt}\left(m\vec{V}\right)_{sys} = \sum\vec{F} = \frac{d}{dt}\int_{CV}\rho\vec{V}\,d\forall + \int_{CS}\rho\vec{V}\left(\vec{v}\cdot\vec{n}\right)dA$$

eq. for a System

eq. for a CV

Let's apply these CV eqs to fluid flow problems — Ch 5

At time t, the same $\sum\vec{F}$ acts on both the CV and the System

$\sum\vec{F}$

t-δt   CV   t+δt $\vec{V}$
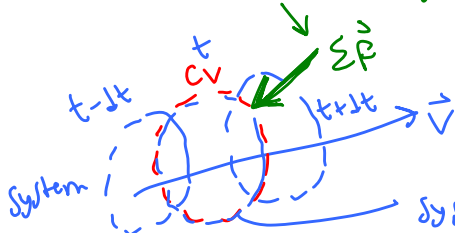
System

System @ time t = <u>SAME</u> as CV @ time t

NOTE: AT ANY INSTANT IN TIME, WE CHOOSE THE SYSTEM THAT OCCUPIES THE SAME VOLUME AS OUR CV → SO, ALL FORCES ACTING ON THE CV ALSO ACT ON THE SYSTEM