Today, we will:

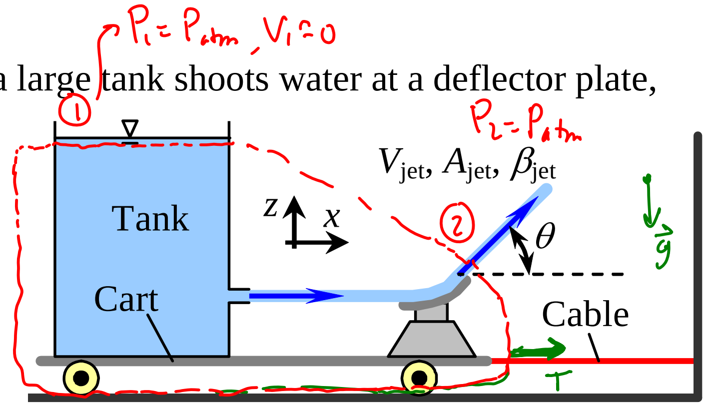
- Do some more example problems – linear CV momentum equation
- Discuss the control volume equation for angular momentum

E. The Linear Momentum Equation for a Control Volume (continued)

4. Examples (continued)

Example: Tension in a cable

Given: A cart with frictionless wheels and a large tank shoots water at a deflector plate, turning it by angle θ as sketched. The cart tries to move to the left, but a cable prevents it from doing so. At the exit of the deflector, the water jet area A_{jet}, its average velocity V_{jet}, and its momentum flux correction factor β_{jet} are known.



To do: Calculate the tension T in the cable.

Solution:

- First step: Pick a CV (a wise one is shown)
- Second step: Use the approximate, most useful form of the linear momentum equation,

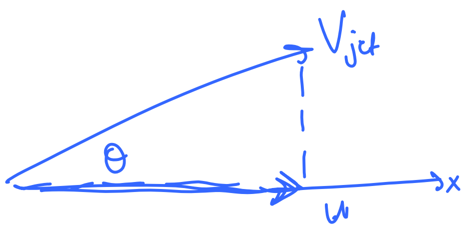
$$\boxed{\sum\vec{F} = \cancel{\sum\vec{F}_{\text{gravity}}} + \cancel{\sum\vec{F}_{\text{pressure}}} + \cancel{\sum\vec{F}_{\text{viscous}}} + \sum\vec{F}_{\text{other}} = \cancel{\frac{d}{dt}\int_{CV}\rho\vec{V}dV} + \sum_{\text{out}}\beta\dot{m}\vec{V} - \cancel{\sum_{\text{in}}\beta\dot{m}\vec{V}}}$$

IN x-direction

none in x; 0, $P = P_{atm}$ all around our CS; (wise CV); $\|$ T (+) x-dir; $=$; quasi-steady; $V \cong 0$ @ (1)

$T = \beta \dot{m} u$ @ (2)

$= \beta_{jet} \rho V_{jet} A_{jet} V_{jet} \cos\theta$

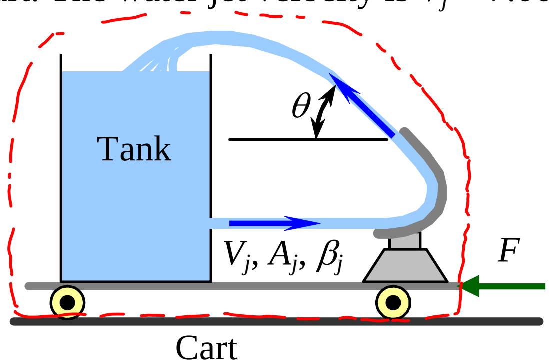


$u = V_{jet} \cos\theta$

$$\boxed{T = \beta_{jet} \rho V_{jet}^2 A_{jet} \cos\theta}$$

Example: Force to hold a cart in place

Given: Water shoots out of a large tank sitting on a cart. The water jet velocity is V_j = 7.00 m/s, its cross-sectional area is A_j = 20.0 mm^2, and the momentum flux correction factor of the jet is 1.04. The water is deflected 135° as shown (θ = 45°), and all of the water flows back into the tank. The density of the water is 1000 kg/m^3.



To do: Neglecting friction on the wheels, calculate the horizontal force F (in units of N) required to hold the cart in place.

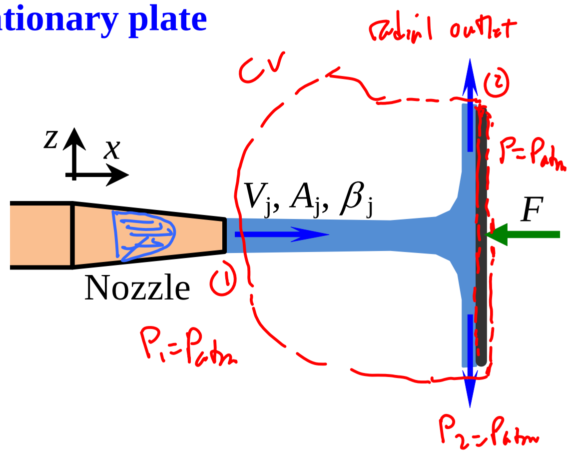
Solution:

- First step: draw a wide CV
- Second step: Use the approximate, most useful form of the linear momentum equation,

$$\boxed{\sum \vec{F} = \cancel{\sum \vec{F}_{\text{gravity}}} + \cancel{\sum \vec{F}_{\text{pressure}}} + \cancel{\sum \vec{F}_{\text{viscous}}} + \sum \vec{F}_{\text{other}} = \cancel{\frac{d}{dt}\int_{CV} \rho \vec{V} dV} + \cancel{\sum_{\text{out}} \beta \dot{m} \vec{V}} - \cancel{\sum_{\text{in}} \beta \dot{m} \vec{V}}}$$

$-F = 0$

Example: Force imparted by a water jet hitting a stationary plate



Given: A horizontal water jet of area A_j, average velocity V_j, and momentum flux correction factor β_j impinges normal to a stationary vertical flat plate.

To do: Calculate the horizontal force F required to keep the plate from moving.

Solution:

- First step: Draw a wise CV
- Second step: Use the approximate, most useful form of the linear momentum equation,

$$\sum \vec{F} = \cancel{\sum \vec{F}_{\text{gravity}}} + \cancel{\sum \vec{F}_{\text{pressure}}} + \cancel{\sum \vec{F}_{\text{viscous}}} + \sum \vec{F}_{\text{other}} = \cancel{\frac{d}{dt}\int_{CV} \rho \vec{V} dV} + \cancel{\sum_{\text{out}} \beta \dot{m} \vec{V}} - \sum_{\text{in}} \beta \dot{m} \vec{V}$$

none in x — cancel $P = P_{atm}$ everywhere — wise CV — $-F$ — steady — $u = 0$ @ outlet (radial flow) — $-\beta \dot{m} u$

x-comp

$$-F = -\beta \dot{m} u$$

$$F = \beta_j \rho V_j A_j V_j$$

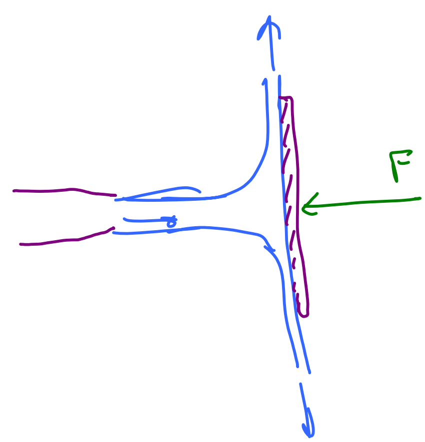
$$\boxed{F = \beta_j \rho V_j^2 A_j}$$

For laminar fully dev. flow @ jet outlet

$V_j = 5.0$ m/s

$A_j = 6.0$ cm^2

$\rho = 1000$ kg/m^3 (water)



Calc F in N

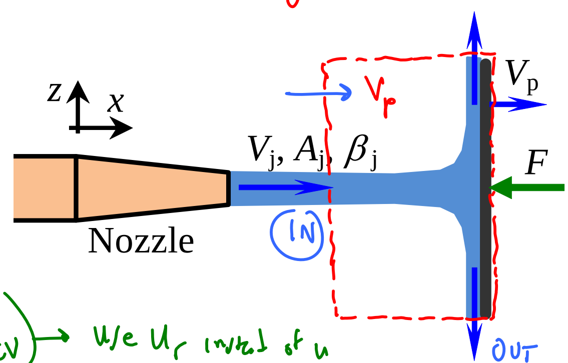
$$F = \beta_j \rho V_j^2 A_j$$

$$= \frac{4}{3}\left(1000 \frac{kg}{m^3}\right)\left(5.0 \frac{m}{s}\right)^2 \left(6.0\ cm^2\right)\left(\frac{N \cdot s^2}{kg \cdot m}\right)\left(\frac{1\ m}{100\ cm}\right)^2$$

$$\boxed{F = 20.0\ N}$$

Example: Force imparted by a water jet hitting a *moving* plate

moving CV



Given: A horizontal water jet of area A_j, average velocity V_j, and momentum flux correction factor β_j impinges normal to a *moving* vertical flat plate. The plate moves to the right at constant speed V_p.

To do: Calculate the horizontal force F required to keep the plate moving at constant speed V_p.

Solution:

- First step: Pick a wise CV
- Second step: Use the approximate, most useful form of the linear momentum equation, in the x-direction, for a moving CV, but *steady*:

(relative to moving CV) → use u_r instead of u

$$\sum F_x = \cancel{\sum F_{x,\,\text{gravity}}} + \cancel{\sum F_{x,\,\text{pressure}}} + \cancel{\sum F_{x,\,\text{viscous}}} + \sum F_{x,\,\text{other}} = \cancel{\sum_{\text{out}} \beta \dot{m} u_r} - \sum_{\text{in}} \beta \dot{m} u_r$$

none in x

$P = P_{atm}$ everywhere

wise CV

$-F$

$u_r = 0$ @ outlet (radial)

relative to the moving CV

@ inlet

$\boxed{u_r = V_j - V_p}$ (relative to the moving CV)

$$-F = -\beta_j \dot{m}_j (V_j - V_p)$$

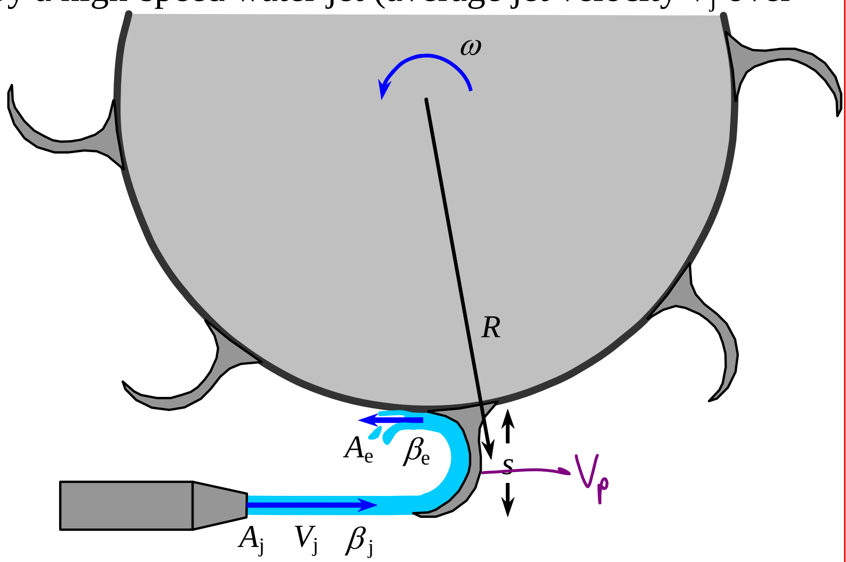
$\dot{m}_j = \rho (V_j - V_p) A_j$

$\rho = \rho_{water}$

$$\boxed{F = \beta_j \rho (V_j - V_p)^2 A_j}$$

Example: Force on a bucket of a Pelton-type (impulse) hydroturbine

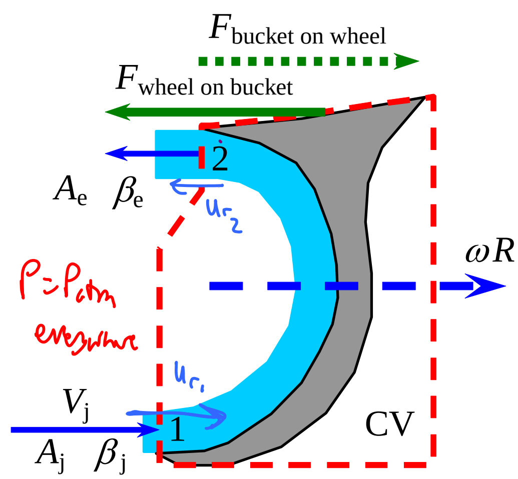
Given: An impulse turbine is driven by a high-speed water jet (average jet velocity V_j over jet area A_j, with momentum flux correction factor β_j) that impinges on turning buckets attached to a turbine wheel as shown. The turbine wheel rotates at angular velocity ω, and is horizontal; therefore, gravity effects are not important in this problem. (The view in the sketch is from the top.) The turning buckets turn the water approximately 180 degrees, and the water exits the bucket over exit cross-sectional area A_e with exit momentum flux correction factor β_e. For simplicity, we approximate that the bucket dimension s is much smaller than turbine wheel radius R ($s << R$).



(**a**) **To do**: Calculate the force of the bucket on the turbine wheel, $F_{\text{bucket on wheel}}$, at the instant in time when the bucket is in the position shown.

(**b**) **To do**: Calculate the power delivered to the turbine wheel.

Solution: We choose a control volume surrounding the bucket, cutting through the water jet at the inlet to the bucket, and cutting through the water exiting the bucket. Note that this is a *moving control volume*, moving to the right at speed ωR. We also cut through the welded joint between the bucket and the turbine wheel, where the force $F_{\text{bucket on wheel}}$ is to be calculated. Because of Newton's third law, the force acting *on the control volume* at this location is equal in magnitude, but opposite in direction, and we call it $F_{\text{wheel on bucket}}$.



Since the pressure through an incompressible jet exposed to atmospheric air is equal to P_{atm}, the pressure at the inlet (1) is equal to P_{atm}, and the pressure at the exit (2) is also equal to P_{atm}.

Solution to be completed in class.

Use relative velocity since this is a moving CV

$$\vec{V}_r = \vec{V} - \vec{V}_{cs}$$

@ ① (inlet) $\vec{V}_r = V_j \vec{i} - \omega R \vec{i}$

in x-dir → $\boxed{u_{r_1} = V_j - \omega R}$

• Use the x-component of the steady linear momentum equation for a moving CV,

$$\sum F_x = \cancel{\sum F_{x,\text{ gravity}}} + \cancel{\sum F_{x,\text{ pressure}}} + \cancel{\sum F_{x,\text{ viscous}}} + \sum F_{x,\text{ other}} = \sum_{\text{out}} \beta \dot{m} u_r - \sum_{\text{in}} \beta \dot{m} u_r$$

none in x $P = P_{atm}$ everywhere   whole CV

$$-F_{\text{bucket on wheel}} = \beta_e \dot{m} u_{r_2} - \beta_j \dot{m} u_{r_1}$$

Cons. of mass → Quasi-steady, relative to our moving CV

$$\sum_{in} \dot{m} = \sum_{out} \dot{m}$$

$$\rho V_{r,in} A_{in} = \rho V_{r_{out}} A_{out}$$

$$\cancel{\rho}\, u_{r,in} A_{in} = \cancel{\rho}\, V_{r_{out}} A_{out}$$

$$(V_j - \omega R) A_j = V_{r_{out}} A_e$$

$$\boxed{V_{r_{out}} = (V_j - \omega R)\frac{A_j}{A_e}}$$

magnitude of outlet velocity @ (2) relative to moving CV

$$\boxed{u_{r_2} = -(V_j - \omega R)\frac{A_j}{A_e}}$$

$$F_{\text{bucket on wheel}} = \dot{m}\left(\beta_j u_{r_1} - \beta_e u_{r_2}\right)$$

$\dot{m}$ = @ inlet,

$$\dot{m} = \rho (V_j - \omega R) A_j$$

$$\boxed{F_{\text{bucket on wheel}} = \rho (V_j - \omega R)^2 A_j \left(\beta_j + \beta_e \frac{A_j}{A_e}\right)}\ ☆$$

(b) $\dot{W}_{\text{wheel}} = \text{Torque} \times \omega = F_{\text{bucket on wheel}} \cdot R \cdot \omega$

$$\boxed{\dot{W}_{\text{wheel}} = \rho (V_j - \omega R)^2 A_j \left(\beta_j + \beta_e \frac{A_j}{A_e}\right) R \omega}\ ☆$$