Today, we will:

- Briefly discuss the control volume angular momentum equation and do an example
- Discuss dimensional analysis and similarity, and the method of repeating variables

F. Conservation of Angular Momentum

1. Equations and definitions

See derivation in the book, using the Reynolds transport theorem (RTT). We set $B = \vec{H} =$ angular momentum $= \vec{r} \times m\vec{V}$ and $b = B/m = \vec{r} \times \vec{V}$. The result is:

General:
$$\sum \vec{M} = \frac{d}{dt}\int_{CV} (\vec{r} \times \vec{V})\rho \, dV + \int_{CS} (\vec{r} \times \vec{V})\rho(\vec{V}_r \cdot \vec{n})\, dA \qquad (6\text{–}47)$$

(Relative velocity)

which is stated in words as

$$\begin{pmatrix}\text{The sum of all}\\ \text{external moments}\\ \text{acting on a CV}\end{pmatrix} = \begin{pmatrix}\text{The time rate of change}\\ \text{of the angular momentum}\\ \text{of the contents of the CV}\end{pmatrix} + \begin{pmatrix}\text{The net flow rate of}\\ \text{angular momentum}\\ \text{out of the control}\\ \text{surface by mass flow}\end{pmatrix}$$

We simplify the control surface integral for cases in which there are well-defined inlets and outlets, just as we did previously for mass, energy, and momentum. The result is:

$$\sum \vec{M} \cong \frac{d}{dt}\int_{CV} (\vec{r} \times \vec{V})\rho \, dV + \sum_{\text{out}} (\vec{r} \times \dot{m}\vec{V}) - \sum_{\text{in}} (\vec{r} \times \dot{m}\vec{V}) \qquad (6\text{–}50)$$

Note that we cannot define an "angular momentum flux correction factor" like we did previously for the kinetic energy and momentum flux terms. Furthermore, many problems we consider in this course are *steady*. For steady flow, Eq. 6-50 reduces to:

Steady flow:
$$\sum \vec{M} = \sum_{\text{out}} (\vec{r} \times \dot{m}\vec{V}) - \sum_{\text{in}} (\vec{r} \times \dot{m}\vec{V}) \qquad (6\text{–}51)$$

Net moment or torque acting on the control volume by external means = Rate of flow of angular momentum *out* of the control volume by mass flow − Rate of flow of angular momentum *into* the control volume by mass flow

Finally, in many cases, we are concerned about only *one* axis of rotation, and we simplify Eq. 6-51 to a scalar equation,

$$\sum M = \sum_{\text{out}} r\dot{m}V - \sum_{\text{in}} r\dot{m}V \qquad (6\text{–}52)$$

Equation 6-52 is the form of the angular momentum control volume equation that we will most often use, noting that r is the shortest distance (i.e., the *normal* distance) between the point about which moments are taken and the *line of action* of the force or velocity being considered. By convention, *counterclockwise moments are positive*.

2. Examples

See Examples 6-8 and 6-9 in the book. Example 6-8 is discussed in more detail here.

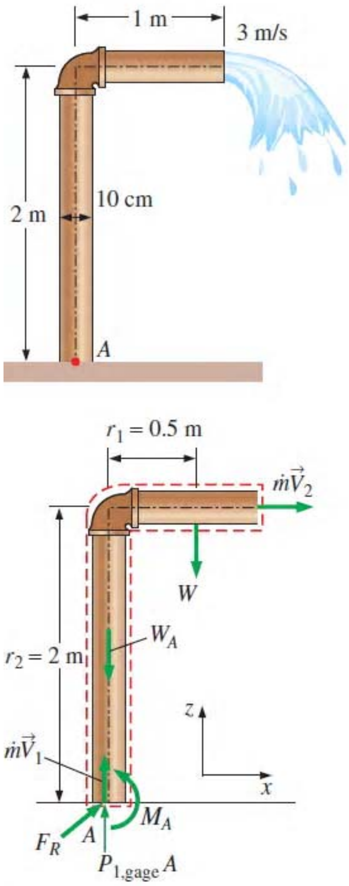


FIGURE 6–39
Schematic for Example 6–8 and the free-body diagram.

EXAMPLE 6–8 Bending Moment Acting at the Base of a Water Pipe

Underground water is pumped through a 10-cm-diameter pipe that consists of a 2-m-long vertical and 1-m-long horizontal section, as shown in Fig. 6–39. Water discharges to atmospheric air at an average velocity of 3 m/s, and the mass of the horizontal pipe section when filled with water is 12 kg per meter length. The pipe is anchored on the ground by a concrete base. Determine the bending moment acting at the base of the pipe (point *A*) and the required length of the horizontal section that would make the moment at point *A* zero.

SOLUTION Water is pumped through a piping section. The moment acting at the base and the required length of the horizontal section to make this moment zero is to be determined.

Assumptions **1** The flow is steady. **2** The water is discharged to the atmosphere, and thus the gage pressure at the outlet is zero. **3** The pipe diameter is small compared to the moment arm, and thus we use average values of radius and velocity at the outlet.

Properties We take the density of water to be 1000 kg/m^3.

Analysis We take the entire L-shaped pipe as the control volume, and designate the inlet by 1 and the outlet by 2. We also take the *x*- and *z*-coordinates as shown. The control volume and the reference frame are fixed.

The conservation of mass equation for this one-inlet, one-outlet, steady-flow system is $\dot{m}_1 = \dot{m}_2 = \dot{m}$, and $V_1 = V_2 = V$ since A_c = constant. The mass flow rate and the weight of the horizontal section of the pipe are

$$\dot{m} = \rho A_c V = (1000 \text{ kg/m}^3)[\pi(0.10 \text{ m})^2/4](3 \text{ m/s}) = 23.56 \text{ kg/s}$$

$$W = mg = (12 \text{ kg/m})(1 \text{ m})(9.81 \text{ m/s}^2)\left(\frac{1 \text{ N}}{1 \text{ kg}\cdot\text{m/s}^2}\right) = 117.7 \text{ N}$$

To determine the moment acting on the pipe at point *A*, we need to take the moment of all forces and momentum flows about that point. This is a steady-flow problem, and all forces and momentum flows are in the same plane. Therefore, the angular momentum equation in this case is expressed as

$$\sum M = \sum_{\text{out}} r\dot{m}V - \sum_{\text{in}} r\dot{m}V$$

where *r* is the average moment arm, *V* is the average speed, all moments in the counterclockwise direction are positive, and all moments in the clockwise direction are negative.

These moments are moments *acting on* the CV.

These moments are moments *due to angular momentum.*

The free-body diagram of the L-shaped pipe is given in Fig. 6–39. Noting that the moments of all forces and momentum flows passing through point *A* are zero, the only force that yields a moment about point *A* is the weight *W* of the horizontal pipe section, and the only momentum flow that yields a moment is the outlet stream (both are negative since both moments are in the clockwise direction). Then the angular momentum equation about point *A* becomes

$$M_A - r_1 W = -r_2 \dot{m} V_2$$

Solving for M_A and substituting give

$$M_A = r_1 W - r_2 \dot{m} V_2$$
$$= (0.5\ \text{m})(118\ \text{N}) - (2\ \text{m})(23.56\ \text{kg/s})(3\ \text{m/s})\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right)$$
$$= \mathbf{-82.5\ N\cdot m}$$

The negative sign indicates that the assumed direction for M_A is wrong and should be reversed. Therefore, a moment of 82.5 N·m acts at the stem of the pipe in the clockwise direction. That is, the concrete base must apply a 82.5 N·m moment on the pipe stem in the clockwise direction to counteract the excess moment caused by the exit stream.

The weight of the horizontal pipe is $w = W/L = 117.7$ N per m length. Therefore, the weight for a length of Lm is Lw with a moment arm of $r_1 = L/2$. Setting $M_A = 0$ and substituting, the length L of the horizontal pipe that would cause the moment at the pipe stem to vanish is determined to be

$$0 = r_1 W - r_2 \dot{m} V_2 \quad \rightarrow \quad 0 = (L/2)Lw - r_2 \dot{m} V_2$$

or

$$L = \sqrt{\frac{2r_2 \dot{m} V_2}{w}} = \sqrt{\frac{2(2\ \text{m})(23.56\ \text{kg/s})(3\ \text{m/s})}{117.7\ \text{N/m}}\left(\frac{\text{N}}{\text{kg}\cdot\text{m/s}^2}\right)} = \mathbf{1.55\ m}$$

Discussion Note that the pipe weight and the momentum of the exit stream cause opposing moments at point A. This example shows the importance of accounting for the moments of momentums of flow streams when performing a dynamic analysis and evaluating the stresses in pipe materials at critical cross sections.

IV. DIMENSIONAL ANALYSIS AND MODELING (Chapter 7)

A. Primary Dimensions:

{m}	{L}	{t}	{T}	{I}	{C}	{N}
mass,	length,	time,	temperature,	elec. current,	intensity of light,	amount of matter

All other dimensions can be formed by combination of these 7 primary dimensions.

e.g. $\{\text{Force}\} = \{\text{mass} \cdot \text{accel}\} = \left\{m \frac{L}{t^2}\right\}$

Example: Primary dimensions – shear stress, force per unit length, and power

(***a***) **Given**: In fluid mechanics, shear stress τ is expressed in units of N/m^2.

To do: Express the primary dimensions of τ, i.e., write an expression for $\{\tau\}$.

Solution:

$$\{\tau\} = \left\{\frac{\text{Force}}{\text{area}}\right\} = \left\{\frac{mL/t^2}{L^2}\right\} = \left\{\frac{m}{L t^2}\right\} = \left\{m^1 L^1 t^{-2}\right\}$$

(***b***) **Given**: Ray is conducting an experiment in which quantity a has dimensions of force per unit length.

To do: Express the primary dimensions of a, i.e., write an expression for $\{a\}$.

Solution:

$$\{a\} = \left\{\frac{F}{L}\right\} \left\{\frac{mL}{t^2 \cdot L}\right\} = \left\{\frac{m}{t^2}\right\} = \left\{m^1 t^{-2}\right\}$$

(***c***) **Given**: Power $\dot{W}$ has the dimensions of energy per unit time.

To do: Write the dimensions of power in terms of primary dimensions.

Solution:

Energy = Force × distance

Power = Energy / time

$$\{\dot{W}\} = \left\{\frac{F \times L}{t}\right\} = \left\{\frac{mL}{t^2}\frac{L}{t}\right\} = \left\{\frac{mL^2}{t^3}\right\}$$

$$\{\dot{W}\} = \left\{\frac{mL^2}{t^a}\right\}$$

$a = 3$

B. Dimensional Homogeneity

$\boxed{\text{All additive terms in an equation must have the same dimensions.}}$ ☆

e.g. CV mom. eq.

$$\sum \vec{F} = \sum_{out} \beta \dot{m} \vec{V} - \sum_{in} \beta \dot{m} \vec{V}$$

$$\left\{ m \frac{L}{t^2} \right\} \quad \left\{ 1 \cdot \frac{m}{t} \frac{L}{t} \right\} \quad \left\{ m \frac{L}{t^2} \right\} \quad ✓$$

C. Dimensional Analysis & Similarity

1. Purposes of Dim. Anal.

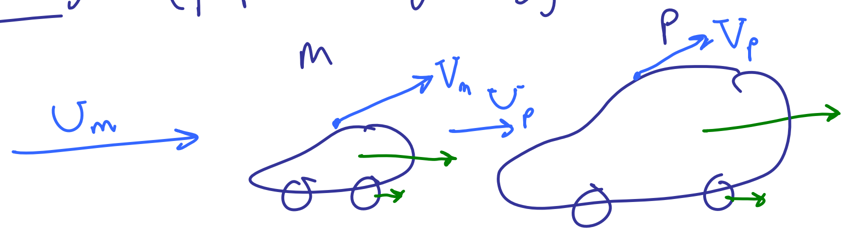
- To help plan & carry out experiments
- To obtain scaling laws [e.g. scale up a small pump to a big pump]
- To (sometimes) predict trends
  - how one variable changes with another variable

2. Similarity → Model vs Prototype

Must have complete similarity to predict performance of the prototype based on model experiments

Criteria for Similarity

1. Geometric similarity (proportional geometry)



2. Kinematic similarity (proportional velocity field)

$$\frac{U_m}{V_m} = \frac{U_p}{V_p}$$

3. Dynamic similarity (proportional forces)

If all 3 hold $\longrightarrow$ total similarity $\rightarrow$ can exactly scale from model to prototype

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3. The Method of Repeating Variables (Buckingham Pi Theorem)

"cookbook" recipe to analyze these problems

3. The Method of Repeating Variables

There are 6 steps that comprise the method of repeating variables. These are listed concisely in Fig. 7-22 in the text, as repeated below:

Step 1: List the parameters in the problem and count their total number n.

Step 2: List the primary dimensions of each of the n parameters.

Step 3: Set the *reduction* j as the number of primary dimensions. Calculate k, the expected number of Π's,

$$k = n - j$$

Π = nondimensional parameter

Step 4: Choose j *repeating parameters.*

Step 4 is often the most difficult or mysterious step. There are guidelines provided in Table 7-3, but it takes practice to know which repeating variables to choose wisely.

Step 5: Construct the k Π's, and manipulate as necessary.

Step 6: Write the final functional relationship and check your algebra.

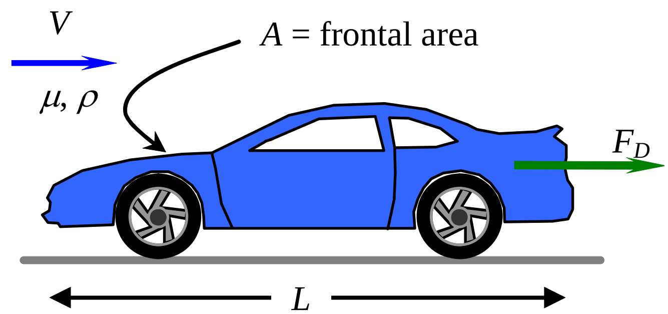
The final functional relationship is given as the *dependent* Π, Π_1, as a function of the *independent* Π's, Π_2, Π_3, … , Π_k, i.e., $\boxed{\Pi_1 = f\left(\Pi_2, \Pi_3, \ldots, \Pi_k\right)}$ ☆ Goal

Guidelines for choosing the repeating variables in Step 4 of the method of repeating variables: (See Table 7-3 in the text for more details):

1. Never pick the *dependent* variable. Otherwise, it may appear in all the Π's, which is undesirable.
2. The chosen repeating parameters must not *by themselves* be able to form a dimensionless group. Otherwise, it would be impossible to generate the rest of the Π's.
3. The chosen repeating parameters must represent *all* the primary dimensions in the problem.
4. Never pick parameters that are already dimensionless. These are Π's already, all by themselves.
5. Never pick two parameters with the *same* dimensions or with dimensions that differ by only an exponent.
6. Whenever possible, choose dimensional constants over dimensional variables so that only *one* Π contains the dimensional variable.
7. Pick common parameters since they may appear in each of the Π's.
8. Pick simple parameters over complex parameters whenever possible.

4. Examples

Example: Dimensional analysis – drag on a car



Given: The drag force F_D on a car is a function of four variables: air velocity V, air density ρ, air viscosity μ, and the length L of the car.

To do: Express this relationship in terms of nondimensional parameters.

Solution: We follow the six steps for the method of repeating variables.

• Step 1 $\quad F_D = \text{fnc}(V, L, \rho, \mu)$ & count $n = 5$

Dependent variable (on left)

• Step 2 $\quad \left\{m \frac{L}{t^2}\right\} \quad \left\{\frac{L}{t}\right\} \quad \{L\} \quad \left\{\frac{m}{L^3}\right\} \quad \left\{\frac{m}{Lt}\right\}$

• Step 3 $\quad$ Guess the reduction j

Usually j = # primary dim's in the problem

Here m, L, t → guess $j = 3$ $\quad$ $j = 3$

[if it does not work out, then set $j = j - 1$ & try again]

The Buckingham Pi Theorem → $\boxed{k = n - j}$ ☆

# of Π's $\quad$ # of original variables $\quad$ reduction

here $k = 5 - 3 = 2$ ← we expect 2 Π's

· Step 4   Choose j repeating parameters   (See Table 7-3)

Pick V, L, ρ

· Step 5   Construct the Π's

$$\Pi_1 = \text{Dependent } \Pi \rightarrow \boxed{\Pi_1 = F_D V^a L^b \rho^c}$$

force Π_1 to be dimensionless

$$\{\Pi_1\} = \{1\} = \{m^0 L^0 t^0\} = \{F_D V^a L^b \rho^c\}$$

$$\{m^0 L^0 t^0\} = \left\{\left(m\frac{L}{t^2}\right)\left(\frac{L}{t}\right)^a (L)^b \left(\frac{m}{L^3}\right)^c\right\}$$

$$m: \quad m^0 = m \cdot m^c = m^{1+c} \rightarrow 0 = 1 + c \rightarrow \boxed{c = -1}$$

$$L: \quad L^0 = L\, L^a L^b L^{-3c} \rightarrow L^0 = L^{1+a+b-3c}$$

$$\boxed{0 = 1 + a + b - 3c}$$

solve for b

$$t: \quad t^0 = t^{-2} t^{-a} \rightarrow 0 = -2 - a \rightarrow \boxed{a = -2}$$

$$0 = 1 - 2 + b + 3 \rightarrow \boxed{b = -2}$$

$$\Pi_1 = F_D V^{-2} L^{-2} \rho^{-1} \rightarrow \boxed{\Pi_1 = \frac{F_D}{\rho V^2 L^2}}$$

"Manipulate" or modify the Π as necessary to agree (if possible) to one of the named, established Π's

(Table 7-4)

Let $\Pi_{1\,\text{modified}} = \boxed{C_D = \text{drag coeff.} = \dfrac{F_D}{\frac{1}{2}\rho V^2 A}}$

more socially acceptable than our original Π

Step 5 (continued) – We repeat this process to construct all the remaining Π's. Here we have only one more, i.e., Π_2

Construct Π_2 using the other (non-used) original variable, μ

$$\boxed{\Pi_2 = \mu V^a L^b \rho^c}$$

$$\{\Pi_2\} = 1 = \{m^0 L^0 t^0\} = \left\{\frac{m}{Lt}\right\}\left(\frac{L}{t}\right)^a (L)^b \left(\frac{m}{L^3}\right)^c$$

$$\{m^0 L^0 t^0\} = \left\{m^1 L^1 t^{-1} \left(L^a t^{-a}\right)\left(L^b\right)\left(m^c L^{-3c}\right)\right\}$$

[This is another way to write it all out, which some people find makes the algebra simpler]

As previously, we equate exponents:

m: $m^0 = m^1 m^c \rightarrow 0 = 1 + c \rightarrow \boxed{c = -1}$

t: $t^0 = t^{-1} t^{-a} \rightarrow 0 = -1 - a \rightarrow \boxed{a = -1}$

L: $L^0 = L^{-1} L^a L^b L^{-3c} \rightarrow 0 = -1 + a + b - 3c$

$$b = 1 - a + 3c = 1 + 1 - 3 = -1$$

$$\boxed{b = -1}$$

So, $\Pi_2 = \mu V^{-1} L^{-1} \rho^{-1} \rightarrow \boxed{\Pi_2 = \frac{\mu}{\rho V L}}$

Table 7-4 $\rightarrow$ If we take Π_2^{-1} we get $\frac{\rho V L}{\mu}$

which is the Reynolds number, Re

So $\boxed{\Pi_{2_{\text{modified}}} = \frac{\rho V L}{\mu} = Re}$

Step 6 $\rightarrow$ Write out the final functional form:

$$\Pi_1 = \text{fnc}(\Pi_2, \Pi_3, \Pi_4 \ldots)$$

Here, $\Pi_1 = \text{fnc}(\Pi_2)$

or $\boxed{C_D = \text{fnc}(Re)}$ ☆

This is our final answer, a relationship between nondimensional parameters

Guidelines for Manipulating the Π Parameters

There are several guidelines for manipulating the Π parameters. These guidelines are listed concisely in Table 7-4 in the text, as summarized below: See Table 7-4 for more details.

1. We may impose a constant (dimensionless) exponent on a Π or perform a functional operation on a Π.
2. We may multiply a Π by a pure (dimensionless) constant.
3. We may form a product (or quotient) of any Π with any other Π in the problem to replace one of the Π's.
4. We may use any of guidelines 1 to 3 in combination.
5. We may substitute a dimensional parameter in the Π with other parameter(s) of the same dimensions.

Substitute A (area) for L^2

The goal is to get each Π into a form that looks like one of the common *named, established* nondimensional parameters that are listed in Table 7-5 in the text. Some of the most popular and often-used ones are listed below. A more exhaustive list is given in the text.

Name	Definition	Ratio of Significance
Darcy friction factor	$f = \dfrac{8\tau_w}{\rho V^2}$	$\dfrac{\text{Wall friction force}}{\text{Inertial force}}$
Drag coefficient	$C_D = \dfrac{F_D}{\frac{1}{2}\rho V^2 A}$ like my π_1	$\dfrac{\text{Drag force}}{\text{Dynamic force}}$
Froude number	$\mathrm{Fr} = \dfrac{V}{\sqrt{gL}} \left(\text{sometimes } \dfrac{V^2}{gL}\right)$	$\dfrac{\text{Inertial force}}{\text{Gravitational force}}$
Lift coefficient	$C_L = \dfrac{F_L}{\frac{1}{2}\rho V^2 A}$	$\dfrac{\text{Lift force}}{\text{Dynamic force}}$
Mach number	$\mathrm{Ma}\ (\text{sometimes } M) = \dfrac{V}{c}$	$\dfrac{\text{Flow speed}}{\text{Speed of sound}}$
Reynolds number	$\mathrm{Re} = \dfrac{\rho V L}{\mu} = \dfrac{VL}{\nu}$	$\dfrac{\text{Inertial force}}{\text{Viscous force}}$

Reynolds number is the most important nondimensional parameter in fluid mechanics.

Name	Definition	Ratio of Significance
Specific heat ratio	$k\ (\text{sometimes } \gamma) = \dfrac{c_p}{c_v}$	$\dfrac{\text{Enthalpy}}{\text{Internal energy}}$
Strouhal number	$\mathrm{St}\ (\text{sometimes S or Sr}) = \dfrac{fL}{V}$	$\dfrac{\text{Characteristic flow time}}{\text{Period of oscillation}}$
Weber number	$\mathrm{We} = \dfrac{\rho V^2 L}{\sigma_s}$	$\dfrac{\text{Inertial force}}{\text{Surface tension force}}$