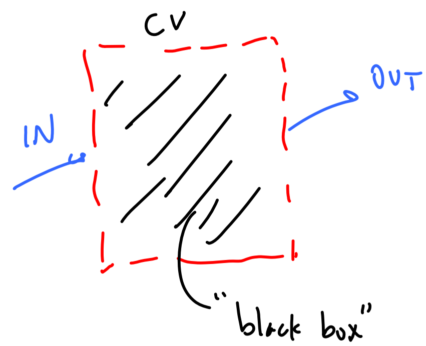
Today, we will:

- Begin our discussion of Chapters 9 and 15 – **Differential Analysis of Fluid Flow**

* NOTE – Read about velocity & volume flow rate meters on your own – see website

VII Differential Analysis

A. Intro – So far we used CV's



Ch. 5, 6, 8

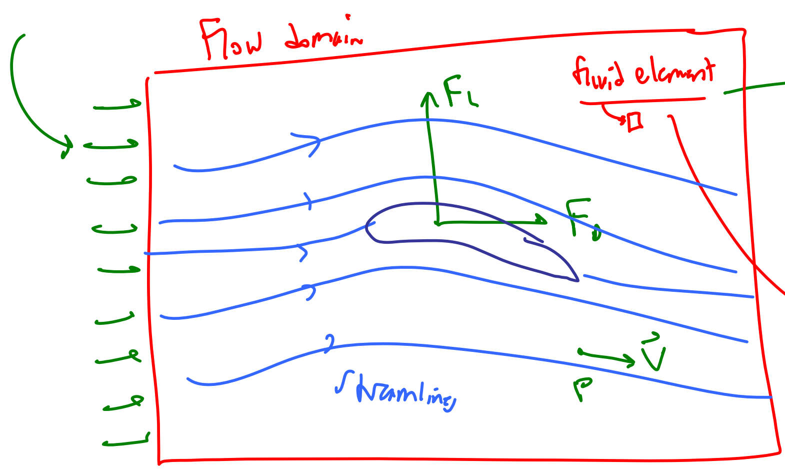
Ch 7 – Dim. Anal.

Now, Ch 9 → Differential analysis

We need to calc. all details inside our flow domain

velocity, pressure, streamlines, etc

BC = Boundary Conditions



Solve for the velocity & pressure fields, everywhere in the flow

Solve differential eq. for a fluid element

We solve the differential eq's:

Ch. 9 → 1) Analytically (pencil & paper) – limited to fairly simple problems

Ch. 15 → 2) Computationally (w/ computers) – CFD

B. Technique of Differential Analysis of Fluid Flow

[nearly the same for analytical & CFD]

Step 1 Identify the flow domain & geometry

Step 2 List assumptions & approximations & BC's

Step 3 List all appropriate eq's & unknowns

E.g. For 3-D incompressible flow w/o significant temperature changes

Unknowns		Equations	
u, v, w	(velocity components)	Cons. of mass	(1)
p	(pressure)	Cons. of mom.	(3)

4 unknowns 4 Eqs

[Add T as unknown #5 & add Energy eq. as eq. #5 if T is significant]

Step 4 Solve Eqs (integrate, solve differential eqs)

Step 5 Apply BCs

IN CFD these two steps are reversed

Step 6 Verify results — • Make sense? ☆

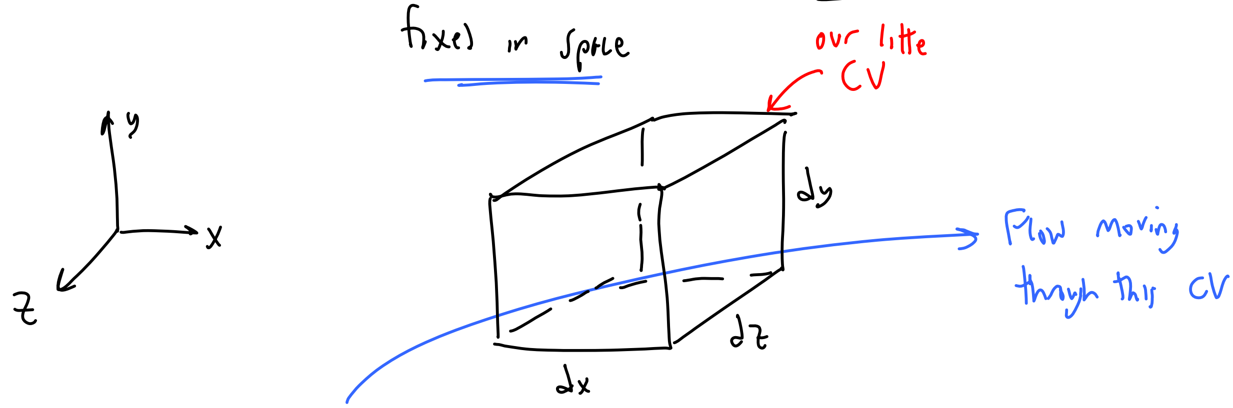
— • Satisfy the BCs?

Equations of Motion

C. Cons. of mass – the continuity eq.

1. Derivation

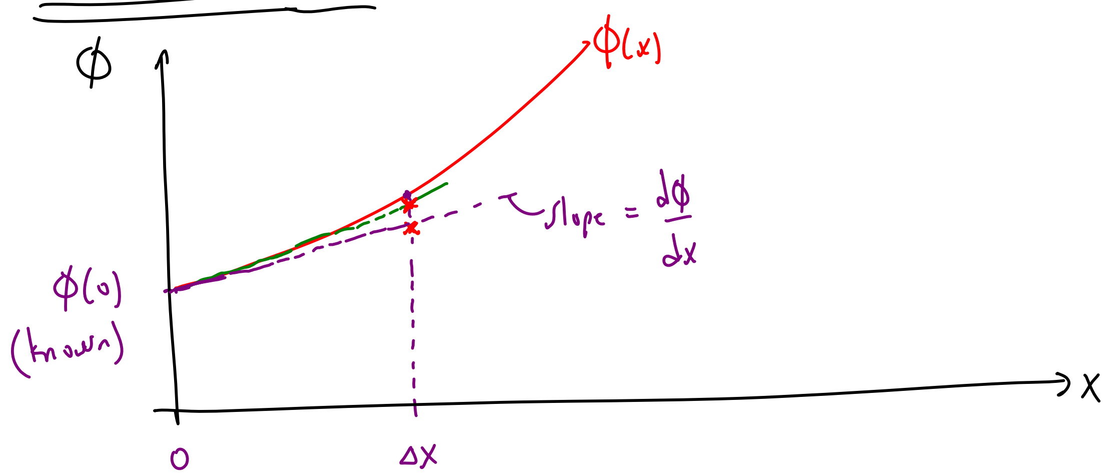
– let's look at a tiny (infinitesimal) control volume fixed in space



Imagine the limit as this CV shrinks to zero size

$$dx, dy, dz \to 0$$

Use Taylor Series Expansions $\qquad \phi = \phi(x)$



For a known $\phi(0)$, we approximate $\phi(\Delta x)$ by a Taylor series expansion:

$$\boxed{\phi(\Delta x) = \phi(0) + \frac{d\phi}{dx}\Delta x + \frac{1}{2!}\frac{d^2\phi}{dx^2}(\Delta x)^2 + \cdots \text{higher-order}}$$

As $\Delta x \to 0$, we can ignore higher-order terms

$$\therefore \quad \boxed{\phi(\Delta x) \approx \phi(0) + \frac{d\phi}{dx}\Delta x}$$

☆ Truncated Taylor Series (truncated to 1st-order)

Derivation of the Continuity Equation
(Section 9-2, Çengel and Cimbala)

We summarize the second derivation in the text – the one that uses a ***differential control volume***. First, we approximate the mass flow rate into or out of each of the six surfaces of the control volume, using ***Taylor series expansions*** around the center point, where the velocity components and density are u, v, w, and ρ. For example, at the right face,

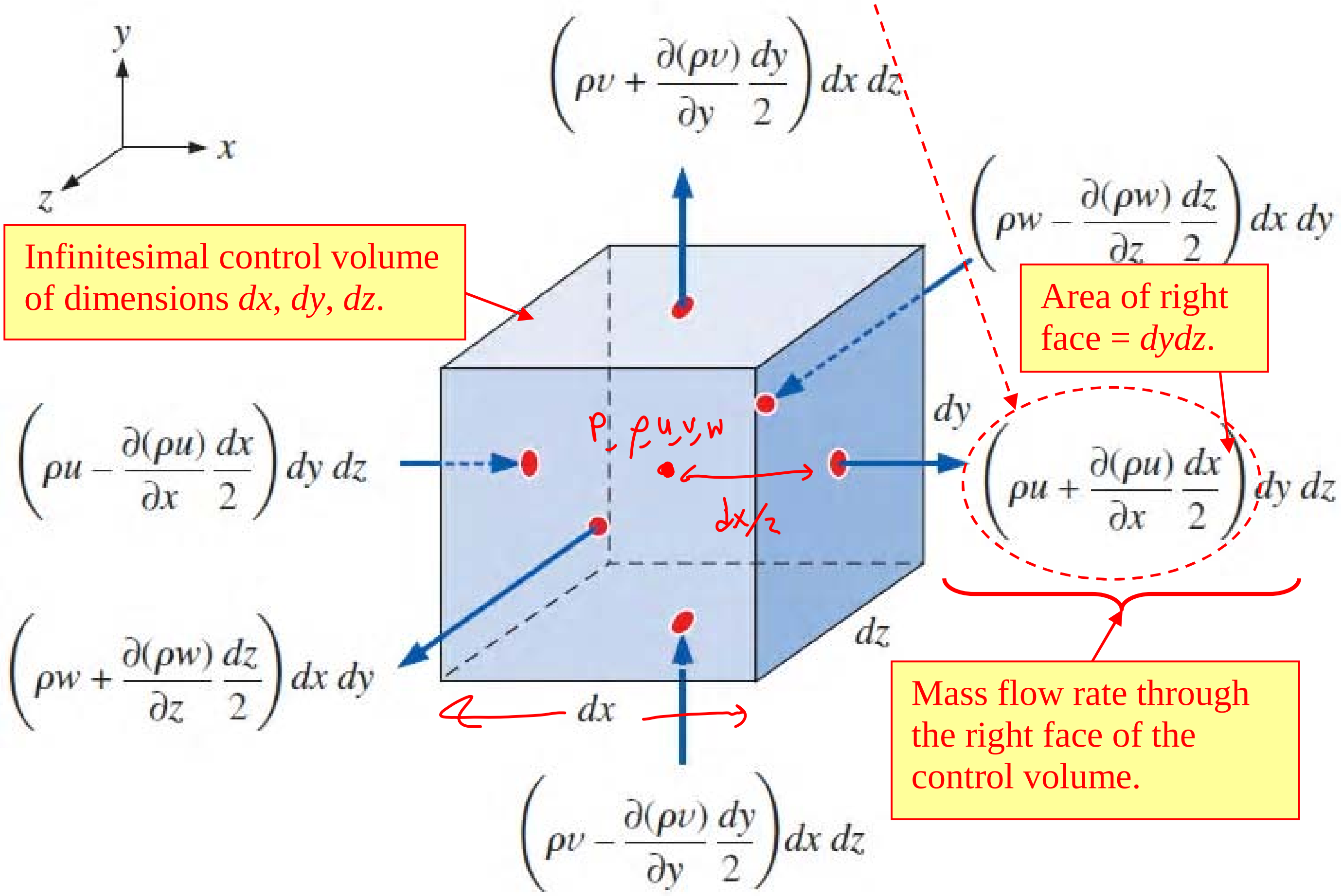
[ρu = mass flow rate per unit area]

del (partial derivative)

Ignore terms higher than order dx.

$$(\rho u)_{\text{center of right face}} = \rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2} + \frac{1}{2!}\frac{\partial^2(\rho u)}{\partial x^2}\left(\frac{dx}{2}\right)^2 + \cdots \qquad \textbf{(9–6)}$$

The mass flow rate through each face is equal to ρ times the normal component of velocity through the face times the area of the face. We show the mass flow rate through all six faces in the diagram below (Figure 9-5 in the text):



Next, we add up all the mass flow rates through all six faces of the control volume in order to generate the general (unsteady, incompressible) ***continuity equation***:

Net mass flow rate into CV: ← all the *positive* mass flow rates (into CV)

$$\sum_{\text{in}} \dot{m} \cong \underbrace{\left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right) dy\,dz}_{\text{left face}} + \underbrace{\left(\rho v - \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\right) dx\,dz}_{\text{bottom face}} + \underbrace{\left(\rho w - \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}\right) dx\,dy}_{\text{rear face}}$$

Net mass flow rate out of CV: ← all the *negative* mass flow rates (out of CV)

$$\sum_{\text{out}} \dot{m} \cong \underbrace{\left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right) dy\,dz}_{\text{right face}} + \underbrace{\left(\rho v + \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\right) dx\,dz}_{\text{top face}} + \underbrace{\left(\rho w + \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}\right) dx\,dy}_{\text{front face}}$$

We plug these into the integral conservation of mass equation for our control volume:

$$\boxed{\int_{\text{CV}} \frac{\partial \rho}{\partial t}\, dV} = \sum_{\text{in}} \dot{m} - \sum_{\text{out}} \dot{m} \qquad \textbf{(9–2)}$$

This term is approximated at the *center* of the tiny control volume, i.e.,

$$\int_{\text{CV}} \frac{\partial \rho}{\partial t}\, dV \cong \frac{\partial \rho}{\partial t}\, dx\,dy\,dz$$

The conservation of mass equation (Eq. 9-2) thus becomes

$$\frac{\partial \rho}{\partial t}\, dx\,dy\,dz = -\frac{\partial(\rho u)}{\partial x}\, dx\,dy\,dz - \frac{\partial(\rho v)}{\partial y}\, dx\,dy\,dz - \frac{\partial(\rho w)}{\partial z}\, dx\,dy\,dz$$

Dividing through by the volume of the control volume, *dxdydz*, yields

Continuity equation in Cartesian coordinates:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0} \qquad \textbf{(9–8)}$$

Finally, we apply the definition of the ***divergence*** of a vector, i.e.,

$$\vec{\nabla} \cdot \vec{G} = \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z} \quad \text{where} \quad \vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \quad \text{and} \quad \vec{G} = \left(G_x, G_y, G_z\right)$$

Letting $\vec{G} = \rho\vec{V}$ in the above equation, where $\vec{V} = (u, v, w)$, Eq. 9-8 is re-written as

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}) = 0 \qquad \textbf{(9–5)}$$

2. Simplifications

• Most general form

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V})=0 \qquad (1)$$

a) Steady but compressible flow

$$\frac{\partial}{\partial t}(\text{anything})=0 \rightarrow \boxed{\vec{\nabla}\cdot(\rho\vec{V})=0} \qquad (2)$$

b) Incompressible but unsteady flow

$$\rho \approx \text{const} \rightarrow \therefore \frac{\partial \rho}{\partial t}=0$$

$$(1) \rightarrow \vec{\nabla}\cdot(\rho\vec{V})=0 \rightarrow \cancel{\rho}\,\vec{\nabla}\cdot(\vec{V})=0$$

$$\boxed{\vec{\nabla}\cdot\vec{V}=0} \qquad (3)$$

Notice → no unsteady term remains!

Eq (3) apply at any instant in time

For incompressible flow, any disturbance is immediately felt everywhere in the flow domain

Incompressible

$[c \rightarrow \infty]$

⊗ The disturbance is felt immediately

c = speed of sound

Compressible

⊗ The disturbance is felt some time later

Eq (3) is our "workhorse" eq's

$$\boxed{\vec{\nabla} \cdot \vec{V} = 0}$$ Incompressible continuity eq.

· Cartesian coord (x, y, z), (u, v, w)

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$$

· Cylindrical coord (r, θ, z), (u_r, u_θ, u_z)

$$\boxed{\frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0}$$

Valid for steady or unsteady incompressible flow

3. Examples

Example: Continuity equation

Given: A velocity field is given by

$$u = a\left(x^2 y + y^2\right)$$
$$v = by^2 x$$
$$w = c$$

To do: Under what conditions is this a valid steady, incompressible velocity field?

Solution:
To be a vaild steady, incompressible velocity field, it must satisfy continuity!

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\cancel{2}a\cancel{x}\cancel{y} + \cancel{2}b\cancel{x}\cancel{y} + 0$$

$$\boxed{a=-b}$$

Example: Continuity equation

Given: A velocity field is given by

$$u = 3x + 4y$$
$$v = by + 2x^2$$
$$w = 0$$

To do: Calculate *b* such that this a valid steady, incompressible velocity field.

Solution:
To be a vaild steady, incompressible velocity field, it must satisfy continuity!

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\underbrace{3 + b}_{} + 0 = 0$$

$$\boxed{b=-3}$$