Today, we will:

- Do more example problems – continuity equation (Cartesian & cylindrical coordinates)
- Discuss the stream function and its physical significance, and do some examples

Example: Continuity equation

Given: A velocity field is given by

$$u = ax + b$$
$$v = \text{unknown}$$
$$w = 0 \rightarrow \text{2-D}$$

To do: Derive an expression for v so that this a valid steady, incompressible velocity field.

Solution: To be a vaild steady, incompressible velocity field, it must satisfy continuity!

$V = V(x,y)$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}} = 0$$

$$a + \frac{\partial v}{\partial y} + 0 \ (\text{2-D}) = 0$$

$\frac{\partial v}{\partial y} = -a \rightarrow$ partial integration $\rightarrow \boxed{v = -ay + f(x)}$

add a fnc of the other variable(s), not just a constant

any function of x

Verify: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = a - a = 0$ ✓

Example: Continuity equation

Given: A flow field is 2-D in the r-θ plane, and its velocity field is given by

$$u_r = \text{unknown}$$
$$u_\theta = c\theta$$
$$u_z = 0$$

To do: Derive an expression for u_r so that this a valid steady, incompressible velocity field.

Solution: To be a vaild steady, incompressible velocity field, it must satisfy continuity!

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \cancel{\frac{\partial u_z}{\partial z}} = 0$$

$$\frac{1}{\cancel{r}}\frac{\partial}{\partial r}(ru_r) + \frac{c}{\cancel{r}} + 0 \ (\text{2-D}) = 0$$

$\times r \rightarrow \frac{\partial}{\partial r}(ru_r) = -c$

Integrate w.r.t. $r \longrightarrow ru_r = -cr + f(\theta)$

$\div r \rightarrow \boxed{u_r = -c + f(\theta)/r}$

Verify:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + 0 = 0$$

$$-\frac{c}{r} + \frac{c}{r} = 0 \ ✓$$

Example: Continuity equation

Given: A flow field is 2-D in the r-θ plane, and its velocity field is given by

$$u_r = -\frac{3}{r} + 2$$
$$u_\theta = 2r + a\theta$$
$$u_z = 0$$

To do: Calculate a such that this a valid steady, incompressible velocity field.

Solution: To be a vaild steady, incompressible velocity field, it must satisfy continuity!

$$\frac{1}{\cancel{r}}\frac{\partial}{\partial r}(ru_r) + \frac{1}{\cancel{r}}\frac{\partial u_\theta}{\partial \theta} + \cancel{\frac{\partial u_z}{\partial z}} = 0$$

2-D

$$\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_\theta}{\partial \theta} = 0$$

$$\underbrace{\frac{\partial}{\partial r}(-3+2r)}_{2} + a = 0$$

$$2 + a = 0 \quad — \quad \boxed{a = -2}$$

Verify on your own

D. The Stream Function, ψ

1. Definition – Consider 2-D, incomp flow in the x-y plane

– Continuity eq. → $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ (1)

Define ψ by its derivatives: $\boxed{u = \frac{\partial \psi}{\partial y} \qquad v = \frac{-\partial \psi}{\partial x}}$

ψ = Stream Function

dimension of ψ are $\frac{L^2}{t}$

Plug into (1) $\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0$

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

order of differentiation does not matter

This is satisfied exactly for any smooth function $\psi(x,y)$

Bottom line → For a given flow, $\Psi(x,y)$ automatically satisfies

Continuity

If we know $\Psi(x,y)$, we can calculate u & v

Example: Stream function

Given: A flow field is 2-D in the *x*-*y* plane, and its stream function is given by

$$\psi(x,y) = ax^3 + byx$$

To do: Calculate the velocity components and verify that this stream function represents a valid steady, incompressible velocity field.

Solution: By definition of the stream function,

$$u = \frac{\partial \psi}{\partial y} = bx \qquad v = -\frac{\partial \psi}{\partial x} = -3ax^2 - by$$

$$\boxed{u = bx} \qquad \boxed{v = -3ax^2 - by}$$

To be a vaild steady, incompressible velocity field, it must satisfy continuity!

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}} = 0$$

$$0 \ (2\text{-D})$$

$$b - b = 0 \ \checkmark$$

Continuity must be satisfied & is automatically satisfied for any smooth $\psi(x,y)$

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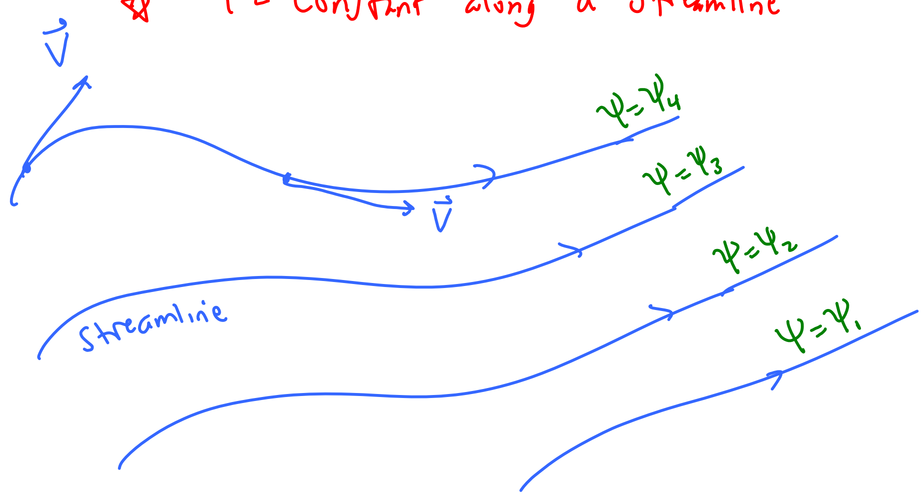
2. Physical Significance of ψ

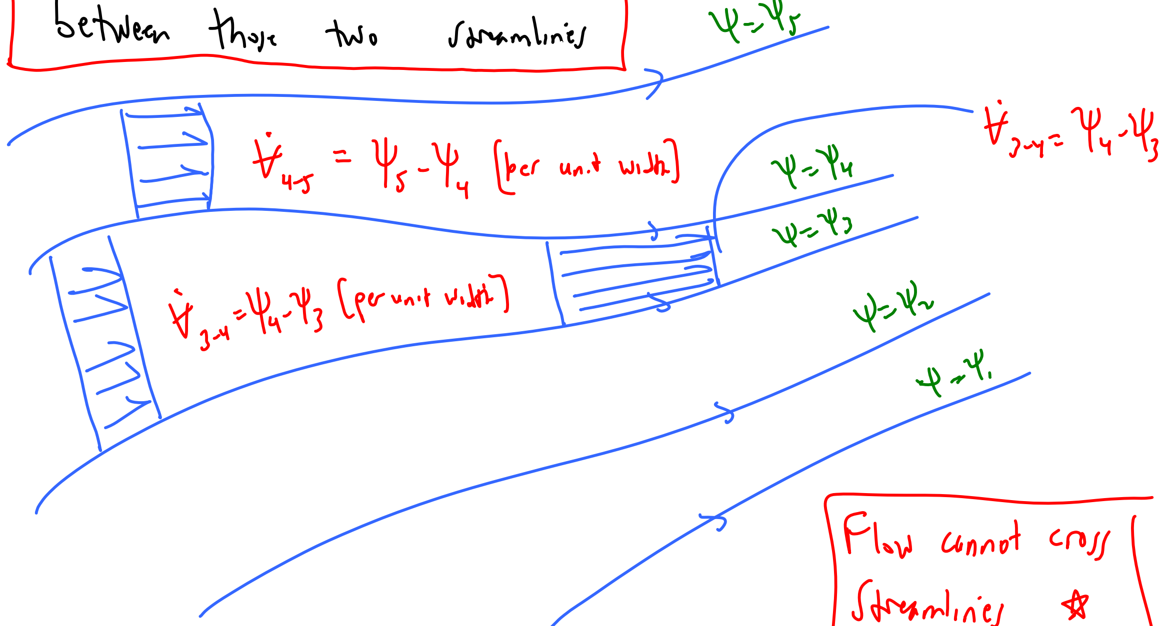
a. Streamlines → $\boxed{\text{Curves of constant } \psi \text{ are streamlines of the flow!}}$

☆ ψ = constant along a streamline



b. Volume flow rate

$\boxed{\text{The difference in } \psi \text{ from one streamline to another is equal to the volume flow rate per unit width (into the page) between those two streamlines}}$



Flow cannot cross streamlines

So, if we set the width as b (into the page)

$$\dot{\forall}_{\text{between } \psi=\psi_4 \ \& \ \psi=\psi_3} = (\psi_4 - \psi_3)\, b$$

$$\frac{m^2}{s} \cdot m = \frac{m^3}{s} \checkmark$$

3. Examples
[See more examples in the textbook.]

Example: Stream function and how to plot streamlines

Given: A steady, 2-D, incompressible flow is given by this velocity field:

$$u = x^2$$
$$v = -2xy - 1$$
$$w = 0$$

To do: Generate an expression for stream function $\psi(x,y)$ and plot some streamlines.

Solution:

First, it is wise to verify that this is a vaild steady, incompressible velocity field. If not, the problem is ill-posed and we could not continue. To verify, it must satisfy continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}} = 0$$

$$2x + -2x = 0 \quad \checkmark$$

0 2-0

Pick $\boxed{\frac{\partial \psi}{\partial y} = u} = x^2$

Partial integration – add function of the other variable

Int. wrt $y \rightarrow$ $\psi = x^2 y + f(x)$

Given v

Now use the other eq for $\psi \rightarrow$ $\boxed{\frac{\partial \psi}{\partial x} = -v} = 2xy + 1$

take x derivative: $\frac{\partial \psi}{\partial x} = \cancel{2xy} + f'(x) = \cancel{2xy} + 1$

$f'(x) = 1$

Int. w.r.t $x \rightarrow f'(x) = 1 \rightarrow f(x) = x + C$

$\boxed{\psi = x^2 y + x + C}$ $\rightarrow$ verify: $u = \frac{\partial \psi}{\partial y} = x^2$

$v = -\frac{\partial \psi}{\partial x} = -2xy - 1$ ✓

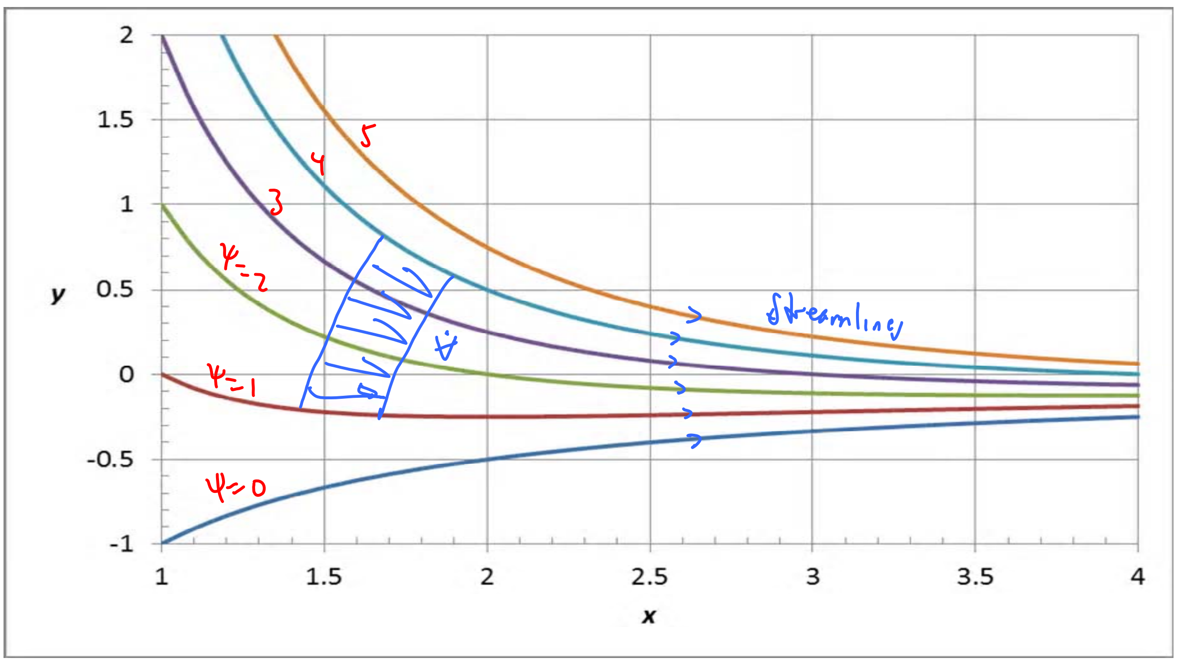
Plot streamlines → $\psi = x^2 y + x + C$ → C is arbitrary, I will set $C = 0$

@ $\psi = \psi_1$ → $\psi_1 = x^2 y + x$ → solve for y:

$$\boxed{y = \frac{\psi_1 - x - c}{x^2}}$$ → Plot this → it will be a streamline

See Excel file on website

Results: for $C = 0$, $\psi = 0, 1, 2, 3, 4, 5$



Example: Streamlines and volume flow rate

Given: The 2-D flow field shown above. The width (into the page) is 0.50 m.

To do: Calculate the volume flow rate (in units of m^3/s) between streamlines $\psi = 1\ m^2/s$ and $\psi = 4\ m^2/s$.

Solution:

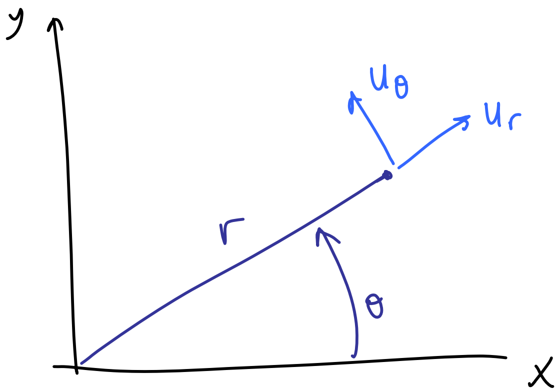
$\psi_4 - \psi_1 = \dot{V}$ per unit width b (into page)

$$\dot{V} = (\psi_4 - \psi_1)\, b = (4-1)\,\frac{m^2}{s}\,(0.5\ m) \rightarrow \boxed{1.5\ m^3/s = \dot{V}}$$

4. Stream function in cylindrical coordinates

2 types

a. Planar flow (x-y plane but we use r, θ coords.) → no z-dependence



Cont. → $\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \cancel{\frac{\partial u_z}{\partial z}} = 0$ 2-D (planar) (1)

Define $\boxed{u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}}$ $\boxed{u_\theta = -\frac{\partial \psi}{\partial r}}$

Verify: plug into (1), this ψ automatically satisfies Eq. (1)

$$\left[\frac{\partial^2 \psi}{\partial r \partial \theta} - \frac{\partial^2 \psi}{\partial \theta \partial r} = 0\right]$$

Works for any smooth func $\psi(r, \theta)$

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b. Axisymmetric flow — in r-z plane no θ dependence

$\psi = \psi(r, z)$ $\boxed{u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}}$ $\boxed{u_z = \frac{1}{r}\frac{\partial \psi}{\partial r}}$