Today, we will:

- Do more examples – the stream function.
- Discuss the differential equation for momentum in fluid flow: **The Navier-Stokes eq**.
- Do some example problems – Navier-Stokes equation

Example: Streamlines in Cylindrical Coordinates

Given: A flow field is steady and 2-D in the r-θ plane, and its stream function is given by

$$\psi = V_\infty r \sin\theta$$

To do:

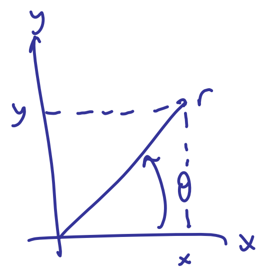
(**a**) Derive expressions for u_r and u_θ and convert to u and v.
(**b**) Sketch the streamlines for this flow field.

Solution:

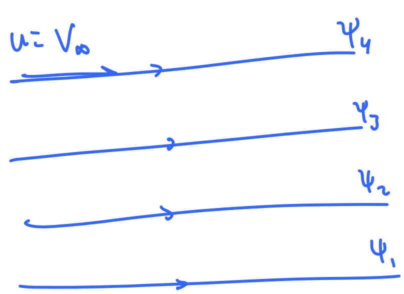
(a) $u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{1}{\cancel{r}} V_\infty \cancel{r} \cos\theta$ $\boxed{u_r = V_\infty \cos\theta}$

$u_\theta = -\frac{\partial \psi}{\partial r} = -V_\infty \sin\theta$ $\boxed{u_\theta = -V_\infty \sin\theta}$

Convert to x,y By definition $\boxed{x = r\cos\theta \quad y = r\sin\theta}$



$\boxed{\psi = V_\infty y}$ → $u = \frac{\partial \psi}{\partial y} = V_\infty$, $v = -\frac{\partial \psi}{\partial x} = 0$ $\boxed{u = V_\infty \quad v = 0}$



($v = 0$)

[b = width into page]

For $V_\infty = 10$ m/s, calculate $\dot{V}/b$ between streamlines @ $y_1 = 0$ m & $y_2 = 5$ m. [ans. in m²/s]

$\frac{\dot{V}}{b} = \psi_2 - \psi_1 = V_\infty y_2 - V_\infty y_1$

$= 10\frac{m}{s}(5\ m) - 0 = \boxed{50\ \frac{m^2}{s}}$

Example: Streamlines in Cylindrical Coordinates

Given: A flow field is steady and 2-D in the r-θ plane, and its velocity field is given by

$$u_r = \frac{c}{r} \qquad u_\theta = 0 \qquad u_z = 0 \rightarrow \text{2-D}$$

To do: Generate an expression for stream function $\psi(r,\theta)$ and plot some streamlines.

Solution:

Pick one eq. for ψ

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{c}{r} \rightarrow \frac{\partial \psi}{\partial \theta} = c \rightarrow \text{Integrate: } \underline{\psi = c\theta + f(r)} \quad (1)$$

Use other eq.:

$$u_\theta = -\frac{\partial \psi}{\partial r} = -f'(r) = 0$$

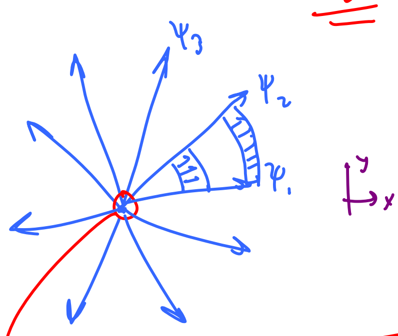
$$f'(r) = 0 \rightarrow \text{int. } \underline{f(r) = \text{const}}$$

Ans. $(1) \rightarrow \boxed{\psi = c\theta + \text{const}}$

const is arbitrary – let's set it to zero

Streamlines are lines of constant ψ = lines of const. θ

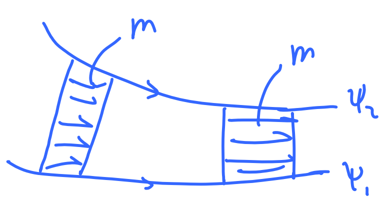
→ rays

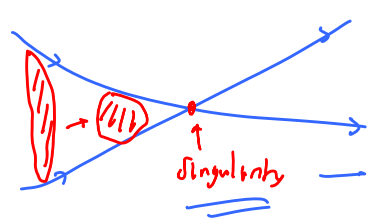


Line source

@ origin, this is a singularity → velocity → ∞

streamlines connect



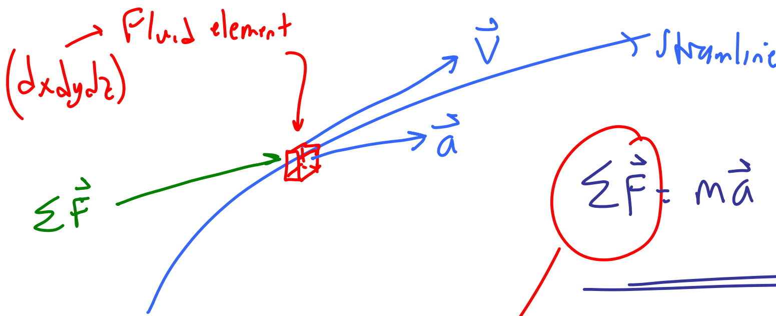


— Not physical, but it is useful as an approximation

E. Cons. of Linear Momentum

1. Derivation See text → 3 separate derivations → all 3 give same result ☺

Easiest derivation → Newton's 2nd law

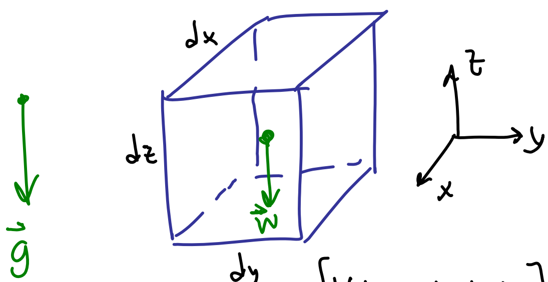


recall, material derivative from Ch. 4

$$\Sigma\vec{F} = m\vec{a} = \rho\, dx\,dy\,dz\, \frac{D\vec{V}}{Dt} \qquad (1)$$

What is $\Sigma\vec{F}$?

$\Sigma\vec{F}$ = body forces + surface forces



[Volume = dxdydz]

$$\Sigma\vec{F} = \underbrace{\rho\vec{g}\, dx\,dy\,dz}_{\vec{W}} + \underbrace{\vec{\nabla}\cdot\sigma_{ij}}_{\text{gives a vector}}\, dx\,dy\,dz$$

Plug into (1):

$$\boxed{\rho\frac{D\vec{V}}{Dt} = \rho\vec{g} + \vec{\nabla}\cdot\sigma_{ij}}$$

☆ CAUCHY's EQ ☆

What is σ_{ij}? — We need to write an expression for σ_{ij} in terms of our primary unknowns, u, v, w, P & fluid properties ρ, μ

We will do this only for Newtonian fluids

Derivation of the Navier-Stokes Equation
(Section 9-5, Çengel and Cimbala)

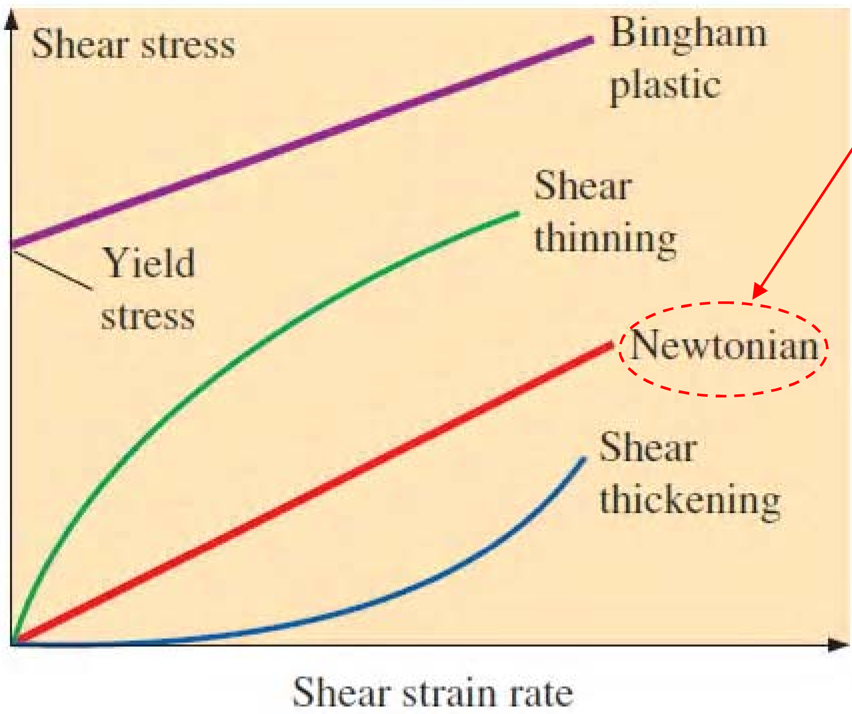
We begin with the general differential equation for conservation of linear momentum, i.e., ***Cauchy's equation***, which is valid for any kind of fluid,

Stress tensor

Cauchy's equation: $$\rho\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}\right] = \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} + \vec{\nabla}\cdot\sigma_{ij}$$

The problem is that the stress tensor σ_{ij} needs to be written in terms of the primary unknowns in the problem in order for Cauchy's equation to be useful to us. The equations that relate σ_{ij} to other variables in the problem – velocity, pressure, and fluid properties – are called ***constitutive equations***. There are different constitutive equations for different kinds of fluids.

Types of fluids:



For ***Newtonian fluids***, *the shear stress is linearly proportional to the shear strain rate.*

Examples of Newtonian fluids: water, air, oil, gasoline, most other common fluids.

recall ε_{ij} from Ch. 4

strain rate tensor

Some examples of non-Newtonian fluids:

- Paint (***shear thinning*** or ***pseudo-plastic***)
- Toothpaste (***Bingham plastic***)
- Quicksand (***shear thickening*** or ***dilatant***).

We consider only Newtonian fluids in this course.

For ***Newtonian fluids*** (see text for derivation), it turns out that

$$\sigma_{ij} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} + \begin{pmatrix} 2\mu \frac{\partial u}{\partial x} & \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & 2\mu \frac{\partial v}{\partial y} & \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & 2\mu \frac{\partial w}{\partial z} \end{pmatrix} \qquad \textbf{(9–57)}$$

We have achieved our goal of writing σ_{ij} in terms of pressure P, velocity components u, v , and w, and fluid viscosity μ.

Now we plug this expression for the stress tensor σ_{ij} into Cauchy's equation. The result is the famous ***Navier-Stokes equation***, shown here for incompressible flow.

Incompressible Navier–Stokes equation:

Navier-Stokes equation:

$$\boxed{\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla} P + \rho \vec{g} + \mu \nabla^2 \vec{V}} \qquad \textbf{(9–60)}$$

To solve fluid flow problems, we need both the continuity equation and the Navier-Stokes equation. Since it is a vector equation, the Navier-Stokes equation is usually split into three components in order to solve fluid flow problems. In Cartesian coordinates,

Incompressible continuity equation:

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0} \qquad \textbf{(9–61a)}$$

x-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \qquad \textbf{(9–61b)}$$

y-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \qquad \textbf{(9–61c)}$$

z-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \qquad \textbf{(9–61d)}$$

2. Applications of the N-S Equation and Examples

Two main applications:

a. Determine the pressure field for a known velocity field
b. Solve fluid flow problems

Example: Stream function and how to plot streamlines

Given: A steady, 2-D, incompressible flow is given by this velocity field:

$$u = x^2 - y^2$$

$$v = -2xy$$

$$w = 0$$

There is no gravity in the *x* or *y* directions (gravity acts only in the *z* direction).

To do: Generate an expression for pressure $P(x,y)$ [the pressure field].

Solution:

First, it is wise to verify that this is a vaild steady, incompressible velocity field. If not, the problem is ill-posed and we could not continue. To verify, it must satisfy continuity:

$$\underbrace{\frac{\partial u}{\partial x}}_{2x} + \underbrace{\frac{\partial v}{\partial y}}_{-2x} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{(2\text{-D})\ +0} = 0 \qquad = 0 \ \checkmark$$

it is a valid incomp. flow

Now plug this velocity field into the Navier-Stokes equation.

$$\boxed{\rho \frac{D\vec{V}}{Dt} = \rho\left[\frac{\partial \vec{V}}{\partial t} + \left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}\right] = -\vec{\nabla}P + \rho\vec{g} + \mu\nabla^2\vec{V}}$$

Let's work on the *x*-component first:

$$\rho\left(\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{\text{steady}} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \underbrace{\cancel{w\frac{\partial u}{\partial z}}}_{2\text{-D}}\right) = -\frac{\partial P}{\partial x} + \underbrace{\cancel{\rho g_x}}_{\text{no g in x-dir}} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \underbrace{\cancel{\frac{\partial^2 u}{\partial z^2}}}_{2\text{-D}}\right)$$

$u = x^2 - y^2$

$\frac{\partial u}{\partial x} = 2x$

$\frac{\partial^2 u}{\partial x^2} = 2$

$$\rho\left((x^2-y^2)(2x) + (-2xy)(-2y)\right) = -\frac{\partial p}{\partial x} + \underbrace{\mu\left(2 + (-2)\right)}_{0}$$

$$\boxed{\frac{\partial p}{\partial x} = -2\rho\left(x^3 + xy^2\right)} \qquad (1)$$

Now let's work on the *y*-component:

$$\rho\left(\underbrace{\cancel{\frac{\partial v}{\partial t}}}_{\text{steady}} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \underbrace{\cancel{w\frac{\partial v}{\partial z}}}_{2\text{-D}}\right) = -\frac{\partial P}{\partial y} + \underbrace{\cancel{\rho g_y}}_{\text{no grav. in y}} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \underbrace{\cancel{\frac{\partial^2 v}{\partial z^2}}}_{2\text{-D}}\right)$$

$$\rho\left[(x^2-y^2)(-2y) + (-2xy)(-2x)\right] = -\frac{\partial p}{\partial y} + \underbrace{\mu\left(0 + 0\right)}_{0}$$

$$\boxed{\frac{\partial p}{\partial y} = 2\rho\left(y^3 + x^2 y\right)} \qquad (2)$$

Check: if $P(x,y)$ is continuous & smooth, $\frac{\partial^2 P}{\partial x \partial y} = \frac{\partial^2 P}{\partial y \partial x}$

verify that this holds for Eqs (1) & (2) ✓

- Pick x-eq. Eq. (1) → integrate wrt x

$$P = -2\rho\left(\frac{x^4}{4} + \frac{x^2y^2}{2}\right) + f(y) \qquad (3)$$

- Take y-der. of this $\frac{\partial P}{\partial y} = -2\rho\left(x^2y\right) + f'(y) = -2\rho\left(y^3 + x^2y\right)$

Compare to Eq (2)

$$f'(y) = -2\rho y^3$$

Int. → $f(y) = -\frac{1}{2}\rho y^4 + C$

Plug into (3):

$$P = -2\rho\left(\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{y^4}{4}\right) + C$$

ANS ☆

$$\boxed{P = -\frac{1}{2}\rho\left(x^4 + 2x^2y^2 + y^4\right) + C}$$

☆ This constant can be anything since P appears in N-S eq. only as derivatives ($\vec{\nabla} p$)

☆ The absolute magnitude of P is not relevant to the flow. Only pressure differences (gradients) matter. ☆