Today, we will:

- Continue Chapter 10 – Approximate solutions of the N-S equation
- Show how to nondimensionalize the N-S equation
- Discuss creeping flow (flow at very low Reynolds number)

B. Nondimensionalization of the Equations of Motion (continued)

Last lecture, we derived the nondimensional form of the continuity equation,

$$\boxed{\vec{\nabla}^* \cdot \vec{V}^* = 0}$$

Now let's do the same thing with the Navier-Stokes equation.

We begin with the differential equation for conservation of linear momentum for a Newtonian fluid, i.e., the ***Navier-Stokes equation***. For incompressible flow,

$$\rho \frac{D\vec{V}}{Dt} = \rho\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}\right] = -\vec{\nabla}P + \rho\vec{g} + \mu\nabla^2\vec{V} \qquad \textbf{(10–2)}$$

Equation 10-2 is *dimensional*, and each variable or property (ρ, $\vec{V}$, t, μ, etc.) is also *dimensional*. What are the primary dimensions (in terms of {m}, {L}, {t}, {T}, etc) of each term in this equation?

A. $\frac{m}{L^2 t}$ B. $\frac{m}{L t^2}$ C. $\frac{m}{L^2 t^2}$ D. $\frac{m}{L t^3}$ E. $\frac{m}{t^2}$

$\{\rho g\} = \left\{\frac{m}{L^3}\,\frac{L}{t^2}\right\} = \left\{\frac{m}{L^2 t^2}\right\}$

Answer: $\left\{\frac{m}{L^2 t^2}\right\}$

To ***nondimensionalize*** Eq. 10-2, we choose ***scaling parameters*** as follows:

TABLE 10–1

Scaling parameters used to nondimensionalize the continuity and momentum equations, along with their primary dimensions

Scaling Parameter	Description	Primary Dimensions
L	Characteristic length	{L}
V	Characteristic speed	$\{Lt^{-1}\}$
f	Characteristic frequency	$\{t^{-1}\}$
$P_0 - P_\infty$	Reference pressure difference	$\{mL^{-1}t^{-2}\}$
g	Gravitational acceleration	$\{Lt^{-2}\}$

We define ***nondimensional variables***, using the scaling parameters in Table 10-1:

$$t^* = ft \qquad \vec{x}^* = \frac{\vec{x}}{L} \qquad \vec{V}^* = \frac{\vec{V}}{V}$$

$$P^* = \frac{P - P_\infty}{P_0 - P_\infty} \qquad \vec{g}^* = \frac{\vec{g}}{g} \qquad \vec{\nabla}^* = L\vec{\nabla} \qquad \text{(10-3)}$$

To plug Eqs. 10-3 into Eq. 10-2, we need to first rearrange the equations in terms of the dimensional variables, i.e.,

$$t = \frac{1}{f} t^* \qquad \vec{x} = L\vec{x}^* \qquad \vec{V} = V\vec{V}^*$$

$$P = P_\infty + \left(P_0 - P_\infty\right) P^* \qquad \vec{g} = g\vec{g}^* \qquad \vec{\nabla} = \frac{1}{L}\vec{\nabla}^*$$

Now we substitute all of the above into Eq. 10-2 to obtain

$$\rho V f \frac{\partial \vec{V}^*}{\partial t^*} + \frac{\rho V^2}{L}\left(\vec{V}^* \cdot \vec{\nabla}^*\right)\vec{V}^* = -\frac{P_0 - P_\infty}{L}\vec{\nabla}^* P^* + \rho g \vec{g}^* + \frac{\mu V}{L^2}\nabla^{*2}\vec{V}^*$$

Every additive term in the above equation has primary dimensions $\{m^1 L^{-2} t^{-2}\}$. To nondimensionalize the equation, we multiply every term by constant $L/(\rho V^2)$, which has primary dimensions $\{m^{-1} L^2 t^2\}$, so that the dimensions cancel. After some rearrangement,

$$\boxed{\left[\frac{fL}{V}\right]}\frac{\partial \vec{V}^*}{\partial t^*} + \left(\vec{V}^* \cdot \vec{\nabla}^*\right)\vec{V}^* = -\boxed{\left[\frac{P_0 - P_\infty}{\rho V^2}\right]}\vec{\nabla}^* P^* + \boxed{\left[\frac{gL}{V^2}\right]}\vec{g}^* + \boxed{\left[\frac{\mu}{\rho V L}\right]}\nabla^{*2}\vec{V}^* \qquad (10\text{–}5)$$

Strouhal number, where $\mathrm{St} = \frac{fL}{V}$

Euler number, where $\mathrm{Eu} = \frac{P_0 - P_\infty}{\rho V^2}$

Inverse of Froude number squared, where $\mathrm{Fr} = \frac{V}{\sqrt{gL}}$

Inverse of Reynolds number, where $\mathrm{Re} = \frac{\rho V L}{\mu}$

Thus, Eq. 10-5 can therefore be written as

Navier-Stokes Equation in Nondimensional Form:

$$[\mathrm{St}]\frac{\partial \vec{V}^*}{\partial t^*} + (\vec{V}^* \cdot \vec{\nabla}^*)\vec{V}^* = -[Eu]\vec{\nabla}^* P^* + \left[\frac{1}{\mathrm{Fr}^2}\right]\vec{g}^* + \left[\frac{1}{\mathrm{Re}}\right]\nabla^{*2}\vec{V}^* \qquad (10\text{–}6)$$

Nondimensionalization **vs.** ***Normalization***:

Equation 10-6 above is *nondimensional*, but not necessarily *normalized*. What is the difference?

- ***Nondimensionalization*** concerns only the ***dimensions*** of the equation – we can use *any* value of scaling parameters *L*, *V*, etc., and we always end up with Eq. 10-6.
- ***Normalization*** is more restrictive than nondimensionalization. To *normalize* the equation, we must choose scaling parameters *L*, *V*, etc. that are appropriate for the flow being analyzed, such that ***all nondimensional variables*** (t^*, $\vec{V}^*$, P^*, etc.) ***in Eq. 10-6 are of order of magnitude unity***. In other words, their minimum and maximum values are reasonably close to 1.0 (e.g., $-6 < P^* < 3$, or $0 < P^* < 11$, but *not* $0 < P^* < 0.001$, or $-200 < P^* < 500$). We express the normalization as follows:

$$t^* \sim 1, \quad \vec{x}^* \sim 1, \quad \vec{V}^* \sim 1, \quad P^* \sim 1, \quad \vec{g}^* \sim 1, \quad \vec{\nabla}^* \sim 1$$

If we have properly normalized the Navier-Stokes equation, we can compare the *relative importance* of various terms in the equation by comparing the *relative magnitudes* of the nondimensional parameters St, Eu, Fr, and Re.

C. The Creeping Flow Approximation

low Reynolds # flow or Stokes flow } when $Re \ll 1$

Re is order of magnitude $\ll 1$ (e.g. 0.05, 0.0032)

$$Re = \frac{\rho V L}{\mu} \ll 1$$ if

- μ very big (e.g. honey)
- V very small (e.g. glaciers)
- L very small (e.g. microorganisms)

(or some combination of these)

NS eq. — steady, incompressible flow w/o gravity effects

$$(\vec{V}\cdot\vec{\nabla})\vec{V} = -\vec{\nabla}P + \mu\nabla^2\vec{V}$$

inertial terms   pressure term   viscous term

Nondimensionalized form:

$$\boxed{\cancel{(\vec{V}^*\cdot\vec{\nabla}^*)\vec{V}^*} = -[Eu]\,\vec{\nabla}^*P^* + \left[\frac{1}{Re}\right]\nabla^{*2}\vec{V}^*} \qquad (1)$$

Creeping flow: $Re \ll 1 \quad \therefore \quad \frac{1}{Re} \gg 1$

if normalized properly, these terms are ~ 1

$\therefore$ $\boxed{Eu \sim \frac{1}{Re} \quad Eu \text{ is big}}$

We can $\therefore$ ignore the 1st term compared to the last term

* We neglect the inertial terms compared to the viscous terms

* Viscous forces are balanced by pressure forces in creeping flow (no inertial forces)

Back to dimensional form of N-S:

Creeping flow eq. of linear momentum:

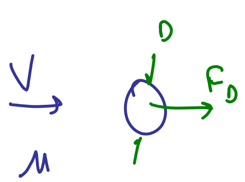
$$\boxed{\vec{\nabla} P = \mu \nabla^2 \vec{V}}$$ ✡ creeping flow approximation

Comments:

- No inertial terms
  - compare human swimming to microorganism swimming
    - human swimming ↓ rely on inertia ↓ we can glide
    - microorganism swimming ↓ no inertia — must keep moving continually to make progress
  - eg squeezing toothpaste
- Density does not appear in the eq

ρ appears only in $Re = \frac{\rho V L}{\mu}$ to see if $Re \ll 1$

eg. Flow over a sphere → Do dimensional analysis [good review]



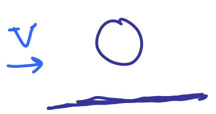
$$F_D = fnc(\mu, V, L) \quad (\text{not a fnc. of } \rho)$$

For creeping flow ⇓

Dim. Anal. $\boxed{F_D = \text{const}\ \mu V L}$ ✡

$$C_{D\,\text{creeping flow}} = \frac{F_D}{\mu V L}$$

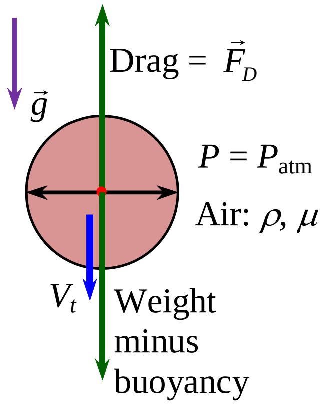
Exact soln → $\boxed{F_D = 3\pi \mu V D}$ ✡



→ larger drag than a sphere

Example: Terminal velocity of a settling air pollution particle

Given: An air pollution particle of diameter 40 microns (40×10^{-6} m) falls towards the ground. After a little while, it reaches **terminal settling velocity** V_t, which is its steady settling velocity in which aerodynamic drag force is balanced by its weight (minus buoyancy). The particle density is 1500 kg/m^3, the air density is 0.840 kg/m^3, and the air viscosity is 1.45×10^{-5} kg/(m s).



To do: Calculate V_t in m/s.

Solution:

$\boxed{F_D = 3\pi\mu V_t D}$

We assume creeping flow, and then will need to check afterwards if the Reynolds number is small enough or not.

$\boxed{\forall_{sphere} = \frac{\pi D^3}{6}}$

Weight = $\rho_p g \forall$ where ρ_p = particle density

Buoyancy = $\rho_a g \forall$ where ρ_a = air density

$F_D = 3\pi\mu V_t D$ (up)

Weight − Buoyancy = $\frac{\pi D^3}{6}(\rho_p - \rho_{air}) g$ (down)

equate

$$18\mu V_t = D^2(\rho_p - \rho_a) g \rightarrow \boxed{V_t = \frac{D^2}{18\mu}(\rho_p - \rho_{air}) g}$$

#s: $$V_t = \frac{(40\times10^{-6}\ \text{m})^2}{18\left(1.45\times10^{-5}\ \frac{\text{kg}}{\text{m}\cdot\text{s}}\right)}(1500 - 0.840)\frac{\text{kg}}{\text{m}^3}\left(9.807\ \frac{\text{m}}{\text{s}^2}\right)$$

$$= V_t = 0.09013\ \text{m/s} \qquad \boxed{V_t = 0.090\ \text{m/s}}$$

Check Re to see if it is $\ll 1$

$$Re = \frac{\rho V_t D}{\mu} = \underline{\underline{0.209}}$$

Is $Re \ll 1$?

Re is small but not very small

Exact answer → $\underline{\underline{V_t = 0.0888\ \text{m/s}}}$

Creeping flow approx. is good to $Re \approx 1$ $(Re < 1)$

D. Approximation for Inviscid Regions of Flow

1. Definition of Inviscid Regions of Flow and the Euler Equation

> Definition: An **inviscid region of flow** is a region of flow in which net viscous forces are negligible compared to pressure and/or inertial forces.

"Inviscid" ≠ no viscosity ! All fluids have viscosity

↓

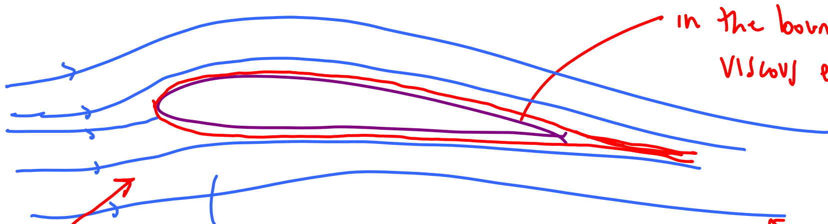
region in which viscous forces are negligible

Creeping flow, $Re \ll 1$ → viscous forces were huge

Inviscid flow $Re \gg 1$ → viscous forces are negligible

Creeping flow & inviscid region of flow are kind of opposite

E.g. Flow over a wing



in the boundary layer (BL) viscous effects are important

Euler eq. is valid here

outside of the BL, viscous effects are not important

For $Re \gg 1$

Let's look at our nondimensionalized Navier-Stokes equation for this case:

$$[\mathrm{St}]\frac{\partial \vec{V}^*}{\partial t^*}+\left(\vec{V}^*\cdot\vec{\nabla}^*\right)\vec{V}^*=-[\mathrm{Eu}]\vec{\nabla}^*P^*+\left[\frac{1}{\mathrm{Fr}^2}\right]\vec{g}^*+\cancel{\left[\frac{1}{\mathrm{Re}}\right]\vec{\nabla}^{*2}\vec{V}^*}$$

$$\boxed{\rho\frac{D\vec{V}}{Dt}=-\vec{\nabla}P+\rho\vec{g}}$$

* Euler Eq.

Inertial terms ~ pressure + gravity

* Euler eq. is valid outside of the Boundary layer, since viscous effects are negligible outside the B.L.