Today, we will:

- Continue discussing inviscid regions of flow; the beloved Bernoulli equation (again)
- Discuss irrotational regions of flow

D. Approximation for Inviscid Regions of Flow (continued)

1. Definition of Inviscid Regions of Flow and the Euler Equation

Definition: An **inviscid region of flow** is a region of flow in which net viscous forces are negligible compared to pressure and/or inertial forces.

We obtained the Euler Equation, Lets consider only steady flow

$$\rho \frac{D\vec{V}}{Dt} = \rho\left[\frac{\partial \vec{V}}{\partial t} + \left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}\right] = -\vec{\nabla}P + \rho\vec{g} \qquad (10\text{-}13)$$

2. The "Beloved" Bernoulli equation in Inviscid Regions of Flow

Recall from Chapter 5, we derived the **Bernoulli equation** as a ***degenerate form of the energy equation*** for cases in which friction and other irreversible losses are negligible,

$$\frac{P}{\rho} + \frac{V^2}{2} + gz = \text{constant along a streamline}$$

It turns out that we can get this *same* equation by working on the Euler equation, and using some vector identities. [See text for derivation, some of which is shown here.]

Vector identity:

$$(\vec{V}\cdot\vec{\nabla})\vec{V} = \vec{\nabla}\left(\frac{V^2}{2}\right) - \vec{V}\times(\vec{\nabla}\times\vec{V}) \qquad \textbf{(10–14)}$$

$\vec{\zeta}$ = vorticity

Recognizing the vorticity vector, Eq. 10-13, the Euler equation, becomes

$\div \rho$

$$\vec{\nabla}\left(\frac{V^2}{2}\right) - \vec{V}\times\vec{\zeta} = -\frac{\vec{\nabla}P}{\rho} + \vec{g} = \vec{\nabla}\left(-\frac{P}{\rho}\right) + \vec{g} \qquad \textbf{(10–15)}$$

where we have divided each term by the density and moved ρ within the gradient operator, since density is constant in an incompressible flow.

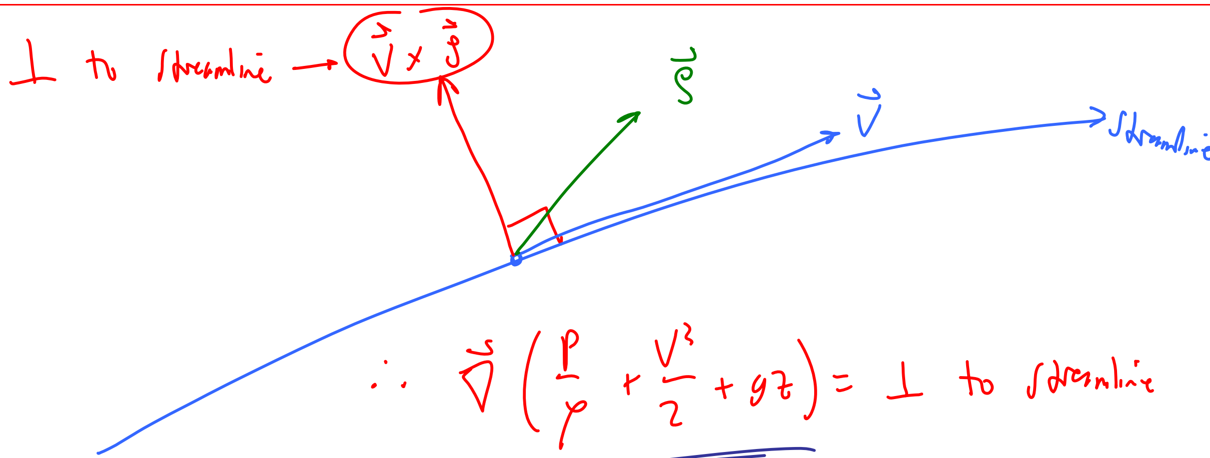
We make the further assumption that gravity acts only in the $-z$-direction (Fig. 10–18), so that

$$\vec{g} = -g\vec{k} = -g\vec{\nabla}z = \vec{\nabla}(-gz) \qquad \textbf{(10–16)}$$

where we have used the fact that the gradient of coordinate z is unit vector $\vec{k}$ in the z-direction. Note also that g is a constant, which allows us to move it (and the negative sign) within the gradient operator. We substitute Eq. 10–16 into Eq. 10–15, and rearrange by combining three terms within one gradient operator,

$$\boxed{\vec{\nabla}\left(\frac{P}{\rho} + \frac{V^2}{2} + gz\right) = \vec{V}\times\vec{\zeta}} \qquad \textbf{(10–17)}$$

Alternate form of steady Euler eq.



$$\therefore \quad \vec{\nabla}\underbrace{\left(\frac{p}{\rho} + \frac{V^2}{2} + gz\right)}_{\text{"something"}} = \perp \text{ to streamline}$$

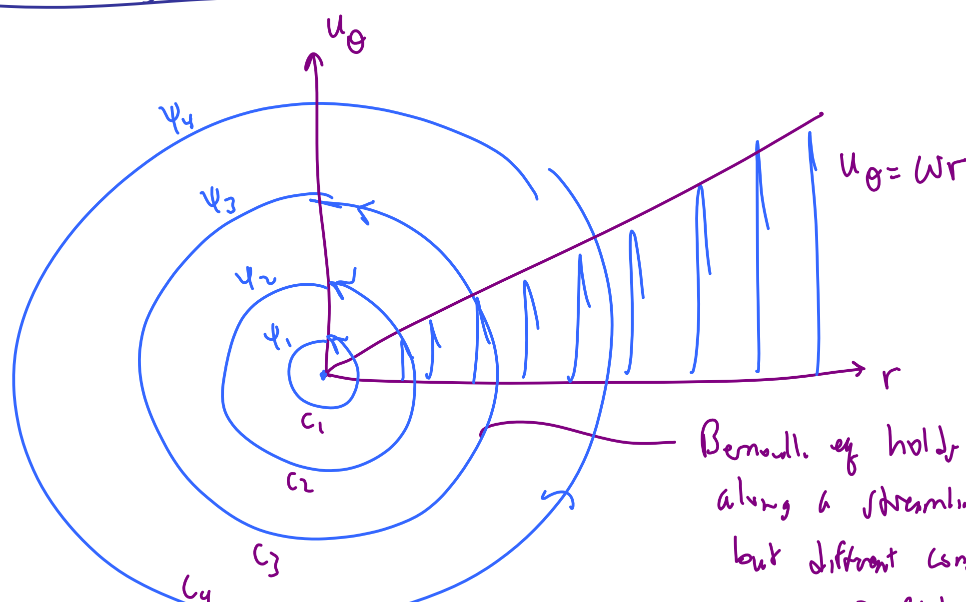
$\vec{\nabla}$(something) is in direction of maximum change of (something)

$\vec{\nabla}$(") = 0 along curves of constant (something)

$$\therefore \quad \boxed{\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant along a streamline} = C_{\infty}}$$ ★

Beloved Bernoulli eq. for inviscid regions of flow

Example: Solid body rotation



Bernoulli eq holds along a streamline but different constant on each streamline

This is an example of a rotational ($\vec{\zeta} \neq 0$) but inviscid flow (no viscous effects)

E. The Irrotational Flow Approximation

1. Intro

Recall, vorticity $= \boxed{\vec{\zeta} = \vec{\nabla} \times \vec{V}}$

if $\vec{\zeta} = 0$, flow is irrotational

if $\vec{\zeta} \neq 0$, flow is rotational

Vector identity: $\boxed{\text{if } \vec{\nabla} \times \vec{V} = 0 \quad \text{then } \vec{V} = \vec{\nabla}\phi}$ ☆ $\vec{V}$ (is any vector)

where ϕ is a scalar called the potential function

Here, let $\vec{V}$ = velocity vector $\vec{V}$

∴ ☆ If the flow is irrotational, then $\vec{\zeta} = \vec{\nabla} \times \vec{V} = 0$,

∴ therefore $\vec{V} = \vec{\nabla}\phi$, where ϕ = velocity potential function

IRROTATIONAL FLOW = POTENTIAL FLOW

Components: $\vec{V} = \vec{\nabla}\phi$

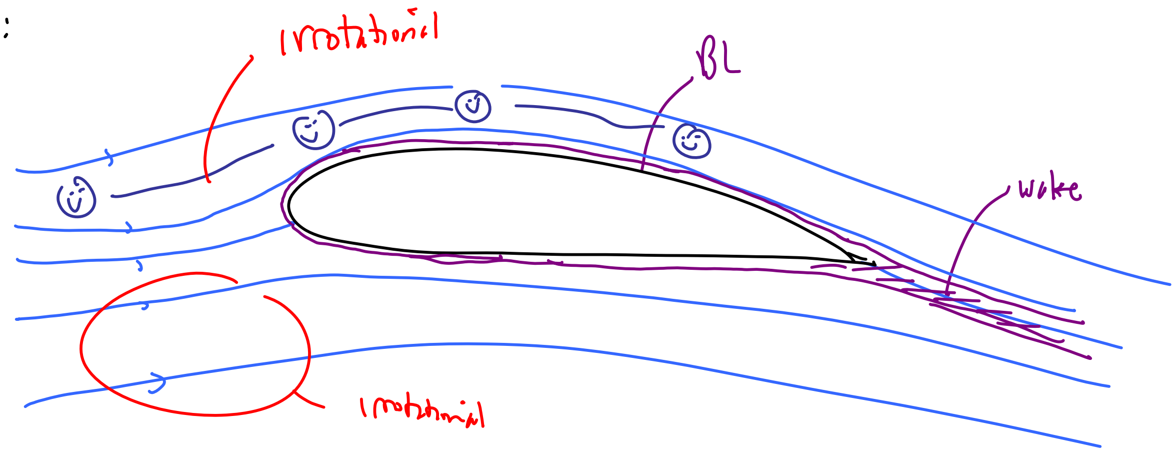
Cartesian coord: $\boxed{u = \dfrac{\partial\phi}{\partial x} \quad v = \dfrac{\partial\phi}{\partial y} \quad w = \dfrac{\partial\phi}{\partial z}}$

Cylindrical ": $\boxed{u_r = \dfrac{\partial\phi}{\partial r} \quad u_\theta = \dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta}, \quad u_z = \dfrac{\partial\phi}{\partial z}}$

For a given scalar ϕ, we know the velocity field

Usefulness:

e.g.



Inside the BL & the wake, $\vec{\zeta} \neq 0$ → flow is rotational

Irrotational flow approx. applies outside of BLs & wakes

2. Eqs of motion for irrotational flow

a. Continuity

$$\vec{\nabla} \cdot \vec{V} = 0$$

but $\vec{V} = \vec{\nabla}\phi$

$$\underbrace{\vec{\nabla} \cdot \vec{\nabla}}\phi = 0$$

$$\nabla^2 \phi = 0$$

$$\boxed{\nabla^2 \phi = 0}$$ ☆ Laplace eq.

e.g. cart.

$$\left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0\right]$$

b. Navier-Stokes

(Steady) →

$$\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla} p + \rho \vec{g} + \mu \nabla^2 \vec{V}$$

$\vec{V} = \vec{\nabla}\phi$

$$\mu \nabla^2 (\vec{\nabla}\phi)$$

$$\mu \vec{\nabla} (\cancel{\nabla^2 \phi})$$

0

(order of differentiation does not matter)

viscous term drops out!

∴ N-S eq. reduces to the Euler Eq.

$$\star \quad \boxed{\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla} P + \rho \vec{g}}$$

Same Euler eq. as we had for inviscid regions of flow, BUT for a different reason → Here it is because the flow is irrotational

→ Previously it was because we neglected viscous terms

C. Bernoulli

When $\vec{\zeta} = 0$ → we can generate the Bernoulli eq. from the Euler Eq. (see text)

$$\star \quad \boxed{\frac{P}{\rho} + \frac{V^2}{2} + gz = C = \text{constant everywhere in the irrotational region of flow}}$$

Irrotational flow is more restrictive than inviscid flow

$\vec{\zeta} = 0$ → ∴ Bernoulli const is const everywhere

$\vec{\zeta}$ may not be zero → Bernoulli const is const only along streamlines

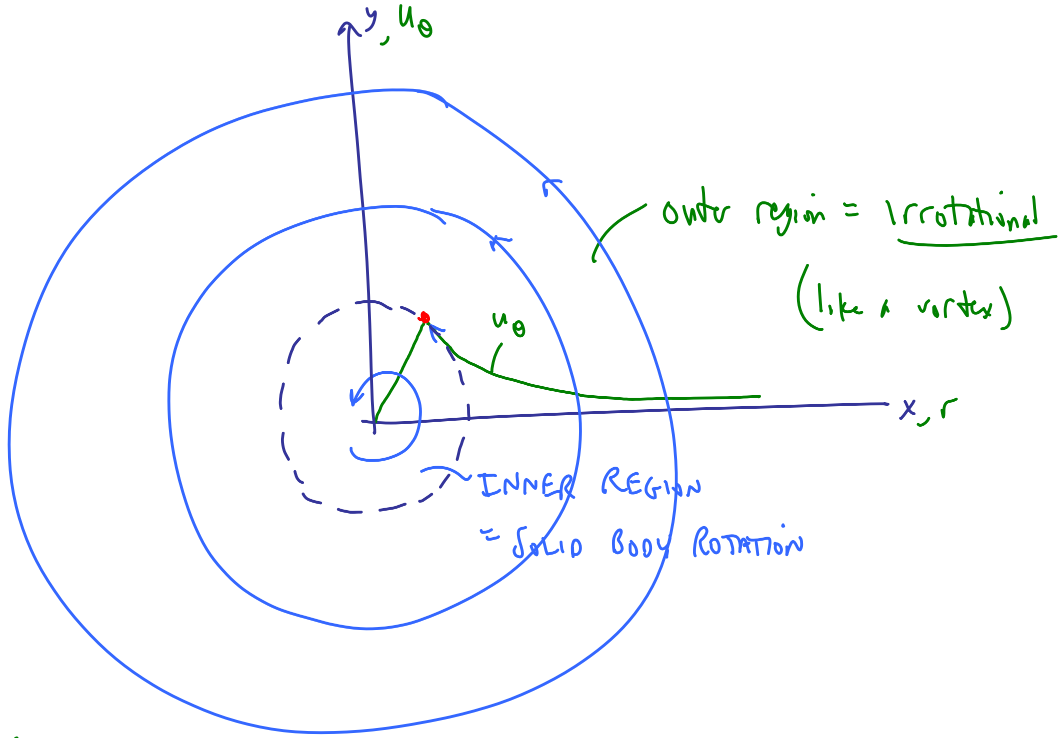
Comment: inviscid regions of flow are not necessarily irrotational

i.e. $\vec{\zeta} \neq 0$. eg. solid body rotation

d. Examples   See Example 10-4 in text

Simple model of a tornado

top view



We apply Bernoulli eq in both regions – but, in inner region, the Bernoulli constant changes as we go to different streamlines

Whereas in the outer region, Bernoulli constant = C = constant everywhere

3. 2-D Irrotational Flow

a. Equations   2-D, steady, incompressible, irrotational flow

- velocity potential $\boxed{\vec{V} = \vec{\nabla}\phi} \rightarrow \boxed{u = \frac{\partial\phi}{\partial x} \qquad v = \frac{\partial\phi}{\partial y}}$

- continuity $\vec{\nabla}\cdot\vec{V} = 0 \rightarrow \boxed{\nabla^2\phi = 0}$ Laplace eq.

$$\boxed{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0}$$

- Stream function   For 2-D flow, $\boxed{u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x}}$

- Irrotationality $\vec{\zeta} = 0 \rightarrow \vec{\nabla} \times \vec{V} = 0$ $\vec{\zeta} = (0, 0, \zeta_z)$

In 2-D, ζ_z component is non-zero, $\zeta_x, \zeta_y = 0$

Plug in ψ

$$\zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad \text{for irrotational flow}$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

$$-\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = 0 \rightarrow -\nabla^2 \psi = 0 \quad \text{or}$$

Laplace again! $\boxed{\nabla^2 \psi = 0}$ ☆

- N-S eq. → reduces to Euler Eq.

But we show in the text that the Euler eq reduces to the beloved Bernoulli eq.

☆ $$\boxed{\frac{P}{\rho} + \frac{V^2}{2} + gz = \text{Constant everywhere}}$$

Summary, equations for 2-D, steady, incompressible, irrotational flow in the *x*-*y* plane:

$\boxed{\vec{\zeta} = \vec{\nabla} \times \vec{V} = 0} \rightarrow \boxed{\vec{V} = \vec{\nabla}\phi} \rightarrow \boxed{\nabla^2 \phi = 0}$; $\boxed{\nabla^2 \psi = 0}$ & $\boxed{\dfrac{P}{\rho} + \dfrac{V^2}{2} + gz = \text{constant everywhere}}$

Cartesian: $\boxed{\nabla^2 \phi = \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0}$, Cylindrical: $\boxed{\nabla^2 \phi = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial \phi}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2} = 0}$

Cartesian: $\boxed{u = \dfrac{\partial \psi}{\partial y}}$ $\boxed{v = -\dfrac{\partial \psi}{\partial x}}$, Cylindrical planar ($r$-$\theta$ plane): $\boxed{u_r = \dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta}}$ $\boxed{u_\theta = -\dfrac{\partial \psi}{\partial r}}$