- Continue discussing the irrotational flow approximation and introduce **superposition**.
- Discuss some elementary planar irrotational flows (building block flows)
- Do some examples of superposition

E. The Irrotational Flow Approximation (continued)

1. Introduction
2. Equations of Motion for Irrotational Flow
3. 2-D Irrotational Flow (continued)
   a. Equations of motion

Summary, equations for 2-D, steady, incompressible, irrotational flow in the *x*-*y* plane:

$$\boxed{\vec{\zeta} = \vec{\nabla} \times \vec{V} = 0} \rightarrow \boxed{\vec{V} = \vec{\nabla}\phi} \rightarrow \boxed{\nabla^2\phi = 0}; \quad \boxed{\nabla^2\psi = 0} \ \& \ \boxed{\frac{P}{\rho} + \frac{V^2}{2} + gz = \text{constant everywhere}}$$

Cartesian: $\boxed{\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0}$, Cylindrical: $\boxed{\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0}$

Cartesian: $\boxed{u = \frac{\partial\psi}{\partial y}}$ $\boxed{v = -\frac{\partial\psi}{\partial x}}$, Cylindrical planar (*r*-θ plane): $\boxed{u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}}$ $\boxed{u_\theta = -\frac{\partial\psi}{\partial r}}$

   b. Superposition

Recall from math class: If a differential eq. is linear, then we can add solutions together, and the sum is also a solution to the diff. eq.

Here, $\nabla^2\phi = 0$ Laplacian is linear → ∴ if ϕ_1 is a soln & ϕ_2 is a soln $\Big\}$ $\phi_3 = C_1\phi_1 + C_2\phi_2$ is also a soln

SUPERPOSITION

Here, $\nabla^2\psi = 0$ → also Laplacian, also linear → $\boxed{\text{Superposition can be applied to both } \phi \text{ and } \psi}$

We can build up complex flows by superposing (adding) simpler flows together

CAUTION!: Superposition works only in irrotational regions of flow – YOU CANNOT USE SUPERPOSITION FOR GENERAL FLOWS

Because the N-S eq. is not linear

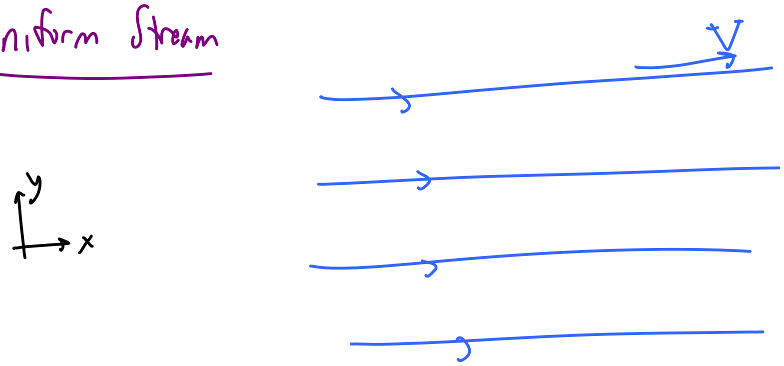
CAUTION 2: The Bernoulli eq. is not linear (V^2 term)

You cannot superpose (add) pressures — you can superpose

- velocity potentials ϕ
- stream functions ψ
- velocity vectors $\vec{V}$

4. Elementary Planar Irrotational Flows [Building Block Flows]

a. Uniform Stream



$u = V = \text{constant}$

$v = 0$

To calc. ψ:

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} = V$$

$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} = 0$$

Integrate

$$\psi = Vy + f(x)$$

$$\frac{\partial \psi}{\partial x} = f'(x) = -v = 0 \rightarrow \text{int.}\ f(x) = \text{Constant}$$

$$\psi = Vy + \text{const}$$

(Constant is arbitrary)

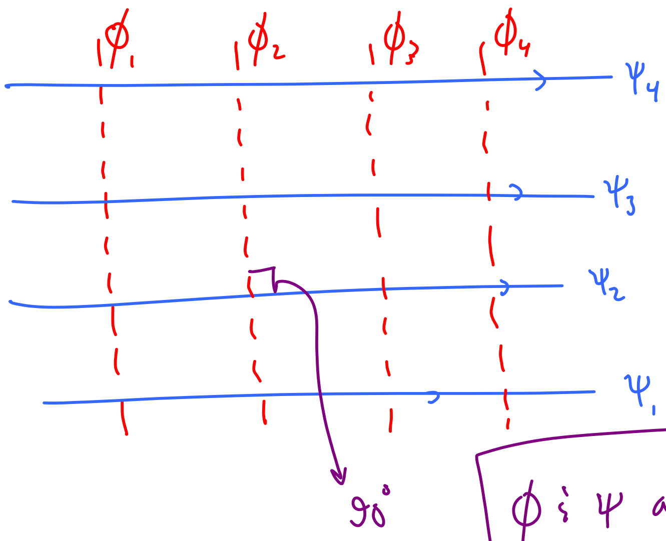
[Set const = 0]

$$\boxed{\psi = Vy} \quad \text{\&} \quad \boxed{\phi = Vx}$$

Similarly

lines of const ψ are streamlines

lines of const ϕ are called equipotential lines

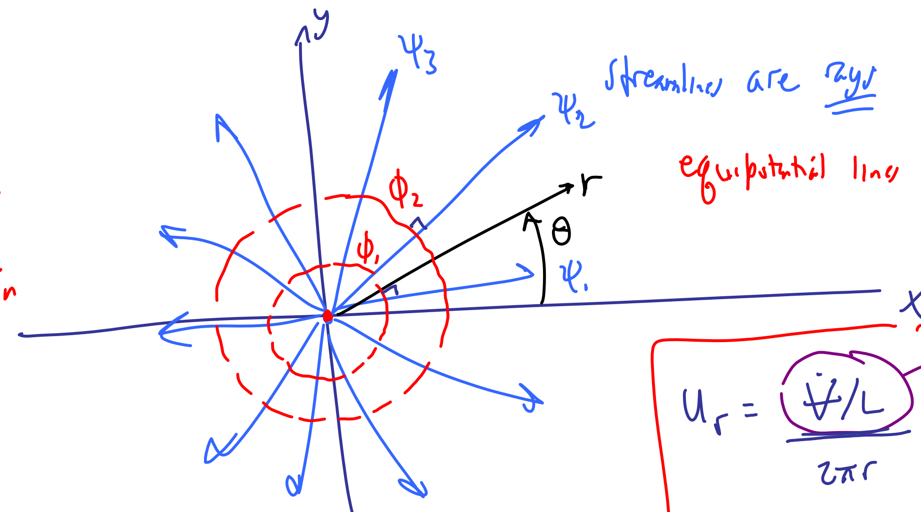


$\boxed{\phi \text{ \& } \psi \text{ are mutually orthogonal}}$

b. Line Source (or sink) (at the origin)

fluid shoots out in all directions along a line // to z-axis

Singularity point @ origin ($V \to \infty$)



streamlines are rays

equipotential lines are circles

$$\boxed{\begin{aligned} u_r &= \frac{\dot{V}/L}{2\pi r} \\ u_\theta &= 0 \end{aligned}}$$

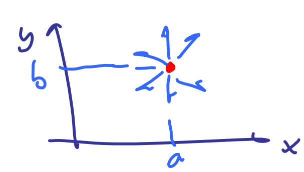
vol. flow rate per unit depth into page

See text → Line source @ origin →

$$\boxed{\begin{aligned} \phi &= \frac{\dot{V}/L}{2\pi} \ln r \\ \psi &= \frac{\dot{V}/L}{2\pi} \theta \end{aligned}}$$

$\boxed{\dot{V}/L = \text{line source strength}}$

- If $\dot{V}/L > 0$ → source. If $\dot{V}/L < 0$, → sink
- Line source @ $(x, y) = (a, b)$ → see text

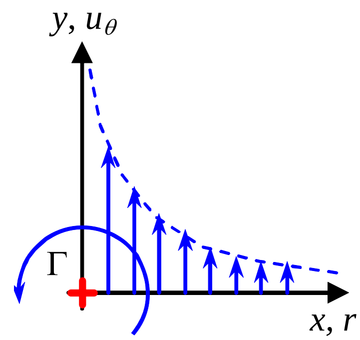


c. Line vortex

Example: Line Vortex – a third building block potential flow

Given: A line vortex around the origin of strength Γ (Γ is also called the **circulation**). The flow is steady and 2-D in the r-θ plane, and its velocity field is given by

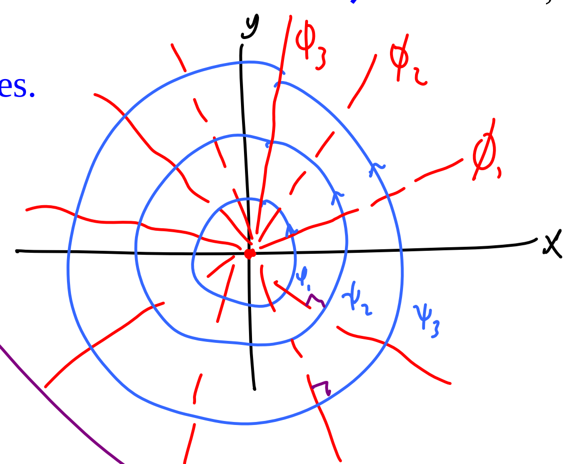
$$\boxed{u_r = 0 \qquad u_\theta = \frac{\Gamma}{2\pi r}}$$



To do: Sketch streamlines and equipotential lines, and generate expressions for ψ and ϕ.

Solution:

A. Streamlines are circles, equipotential lines are circles.
B. Streamlines are circles, equipotential lines are rays.
C. Streamlines are rays, equipotential lines are circles.
D. Streamlines are rays, equipotential lines are rays.



Singularity @ origin

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\Gamma}{2\pi r}$$

$\times r$ $\quad \frac{\partial\phi}{\partial\theta} = \frac{\Gamma}{2\pi}$ → integrate $\quad \phi = \frac{\Gamma}{2\pi}\theta + f(r)$

also $\quad u_r = \frac{\partial\phi}{\partial r} = 0 + f'(r) = 0$ → $f'(r) = 0$ → $f(r) =$ const.

Constant is arbitrary. let's set it to zero

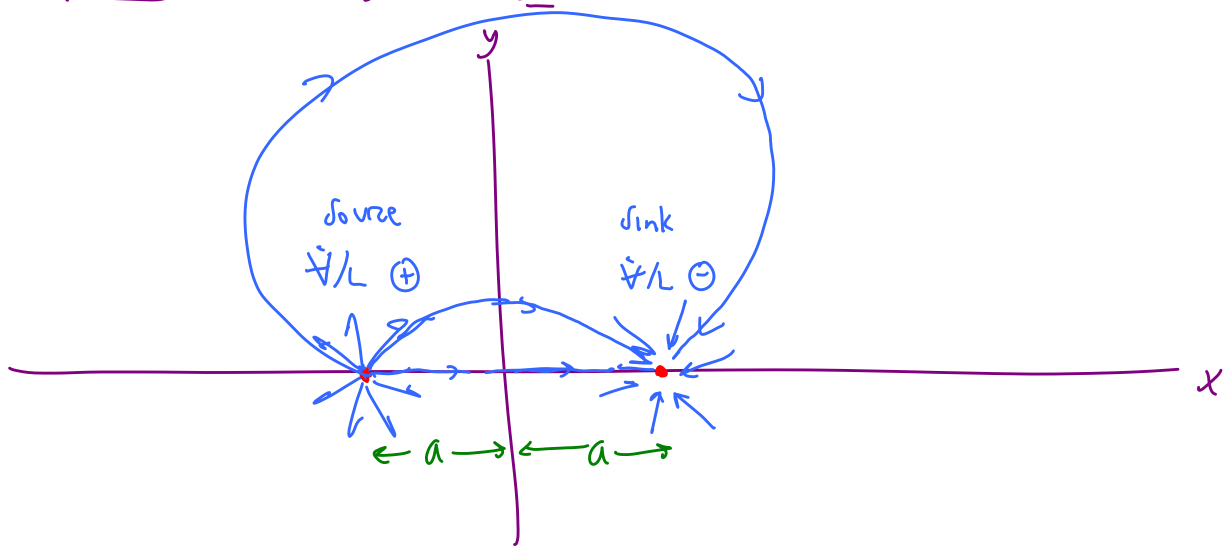
$$\boxed{\phi = \frac{\Gamma}{2\pi}\theta} = \text{rays}$$

similarly, $\quad \boxed{\psi = -\frac{\Gamma}{2\pi}\ln r} = \text{circles}$

} at origin

- $\Gamma > 0$ → counterclockwise vortex (mathematically +)
- $\Gamma < 0$ — clockwise " (" −)
- Line vortex @ (a,b) — shift the origin (see text)

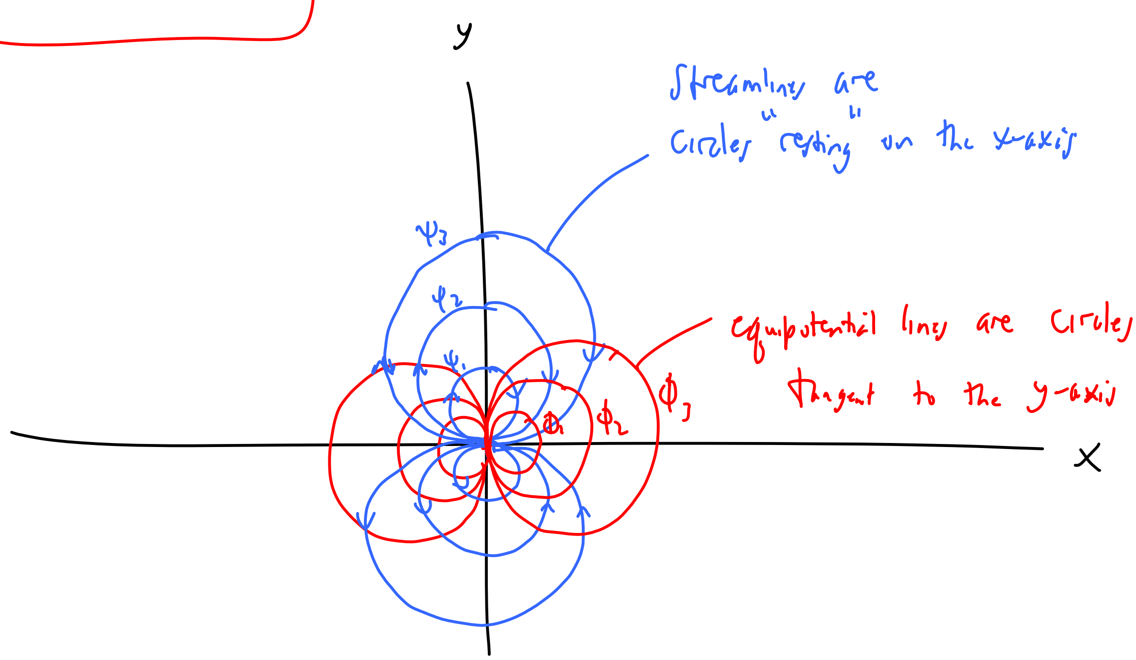
d. <u>Doublet</u> (building block <u>and</u> example of superposition)



Let $a \to 0$ & let $\dot{\forall}/L \to \infty$ simultaneously

Let $\boxed{K = \text{Doublet strength} = \frac{a}{\pi}\frac{\dot{\forall}}{L}}$

get
$$\phi = K\frac{\cos\theta}{r}$$
$$\psi = -K\frac{\sin\theta}{r}$$
→ Doublet @ the origin (origin = singularity)



5. Examples of Superposition

a. The Rankine Half-Body

Superpose

- Uniform stream, V
- Line source @ origin, $\dot{\mathcal{V}}/L$

add up either ψ or ϕ

For ψ,

free stream

source

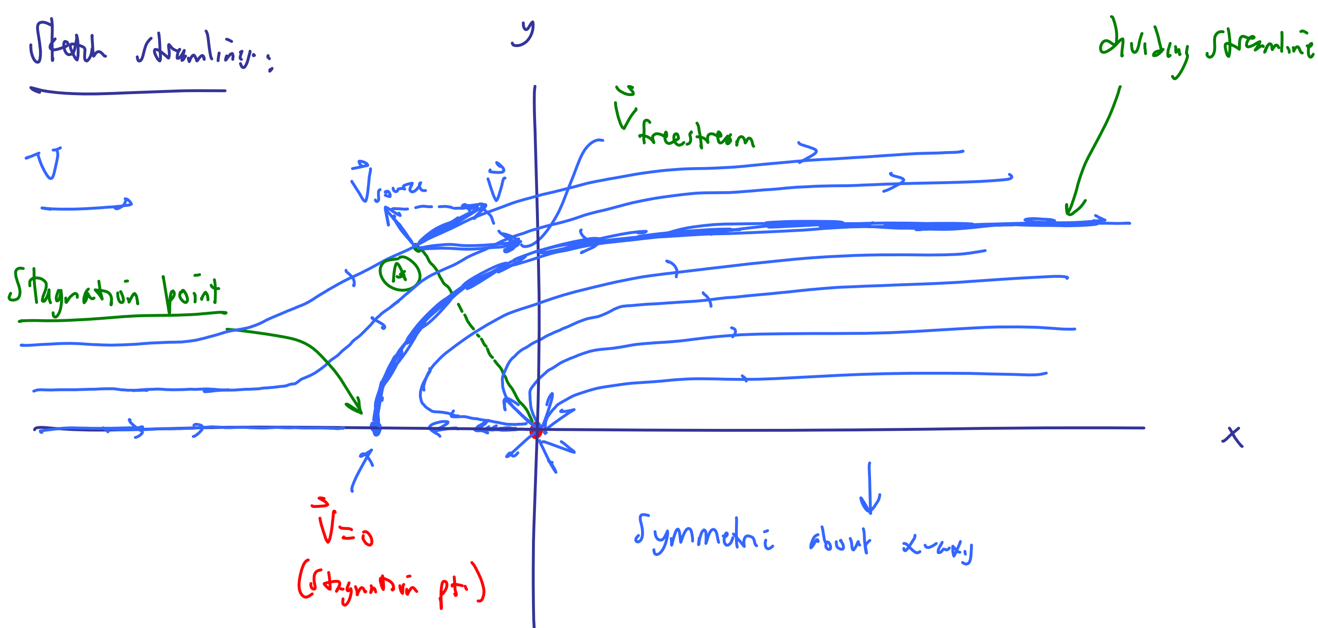
$$\boxed{\psi = Vy + \frac{\dot{\mathcal{V}}/L}{2\pi}\theta}$$

$$\boxed{\psi = Vr\sin\theta + \frac{\dot{\mathcal{V}}/L}{2\pi}\theta}$$

This is a valid irrotational flow

What is it?

Sketch streamlines:



At any location in the flow, the velocity vector is the sum of the velocity vectors of the two building block flows — see pt. (A) above