Today, we will:

- Discuss the BL equations and the BL procedure
- Do a BL example, boundary layer on a flat plate aligned with the flow

F. The Boundary Layer Approximation (continued)

1. Introduction
2. The BL coordinate system
3. The BL equations – see textbook for derivation

These BL equations are valid for steady, laminar, incompressible, 2-D flow in the *x*-*y* plane, and the Reynolds number, $\mathrm{Re}_L = \rho V L/\mu$, must be high enough for the approximations to be valid, but small enough for the flow to remain laminar:

Continuity:

$$\boxed{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0}$$

***x*-momentum:**

$$\boxed{u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}}$$

Or, using Bernoulli in the outer flow:

$$\boxed{u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2}}$$

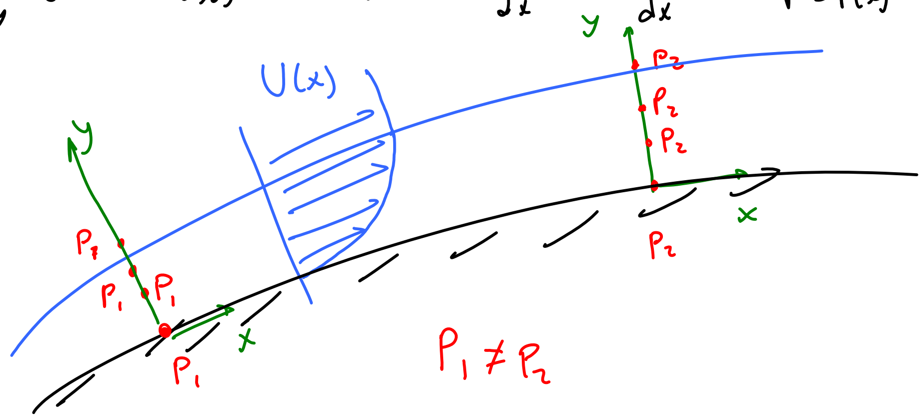
***y*-momentum:**

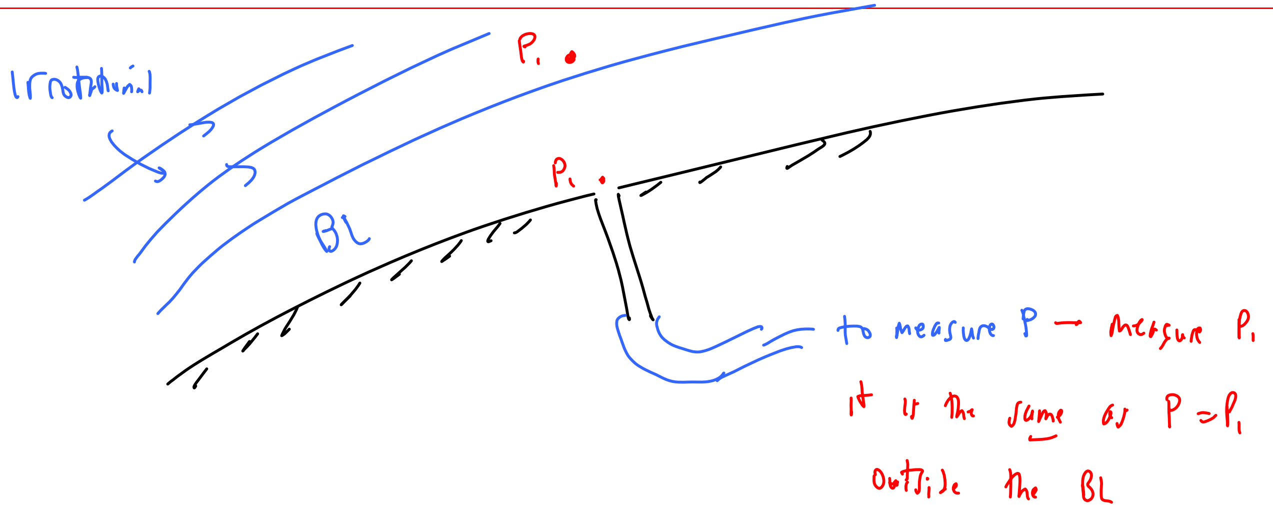
$$\boxed{\frac{\partial P}{\partial y}\approx 0 \ \text{ through the BL}} \rightarrow \boxed{P=P(x) \ \text{ only}}$$

Significant

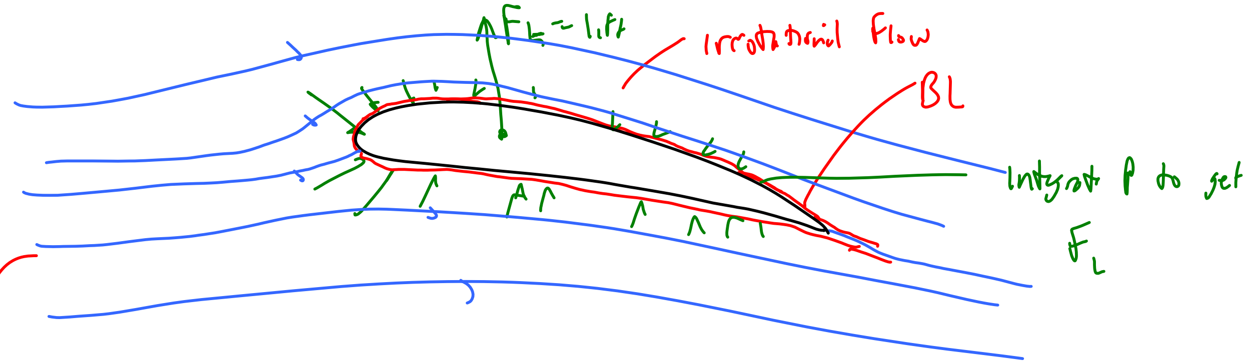
P does not vary across a BL (y) → P varies along a BL (x)

$$\frac{\partial P}{\partial y}\approx 0 \rightarrow P(x,y)\approx P(x) \rightarrow \frac{\partial P}{\partial x} \rightarrow \frac{dP}{dx} \qquad P=P(x)$$





Example → Design of airfoil (wing) (2-D)



We can easily solve for the irrotational outer flow (ignoring the BL)

easier than solving N-S eq.

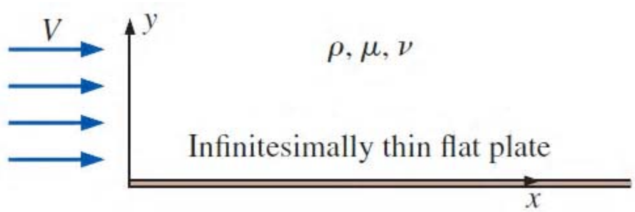
- use Euler Eq. + Bernoulli
- or use potential flow superposition

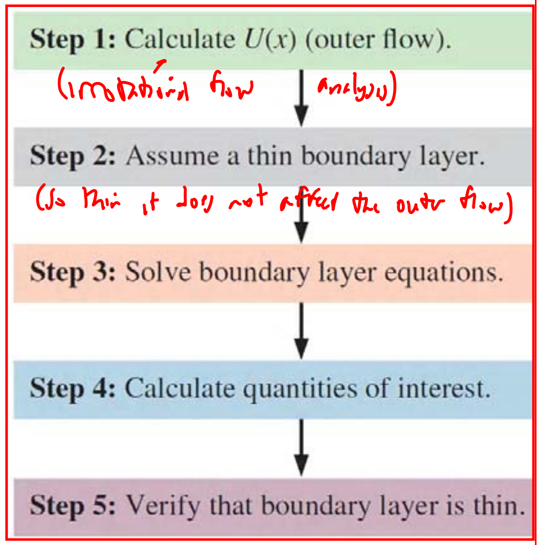
The pressure field obtained by irrotational flow analysis is valid even to the wall (through the BL)

Comment: Potential (irrotational) flow analysis can give us the correct pressure distribution & we can calculate lift very accurately. However, to get the drag on the 2-D wing, we need to add a BL & do a BL analysis using the technique described next.

4. The Boundary Layer Procedure

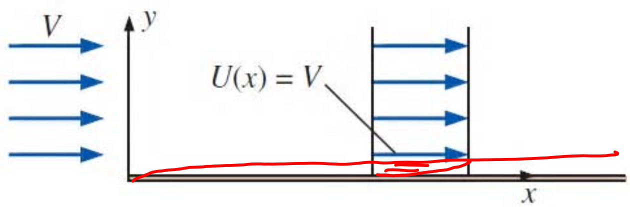
Example: The Laminar Flat Plate Boundary Layer





We go through the steps of the boundary layer procedure:

- **Step 1**: The outer flow is ***U*(*x*) = *U* = *V* = constant**. In other words, the outer flow is simply a uniform stream of constant velocity.
- **Step 2**: A very thin boundary layer is assumed (so thin that it does not affect the outer flow). In other words, the outer flow does not even know that the boundary layer is there.



- **Step 3**: The boundary layer equations must be solved; they reduce to

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0} \qquad \boxed{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}}$$

There are four required boundary conditions,

no-slip $\left\{ \begin{array}{ll} u = 0 & \text{at } y = 0 \\ v = 0 & \text{at } y = 0 \end{array} \right.$ $\quad u = U \quad \text{as } y \to \infty$ @ edge of BL

$u = U \quad \text{for all } y \text{ at } x = 0$ outer flow does not "feel" the plate until $x=0$

This equation set was first solved by **P. R. H. Blasius** in 1908 – numerically, but *by hand*!

Blasius introduced a **similarity variable** η that combines independent variables x and y into one nondimensional independent variable,

Similarity variable

$$\eta = y\sqrt{\frac{U}{\nu x}}$$

and he solved for a nondimensionalized form of the x-component of velocity,

$$f' = \frac{u}{U} = \text{function of } \eta$$

The similarity solution is f' as a function of η.

The key here is that ***one single similarity velocity profile holds for any x-location along the flat plate***. In other words, the velocity profile shape is the same ("similar") at any location, but it is merely *stretched vertically* as the boundary layer grows down the plate. This is illustrated in Fig. 10-98 in the text.

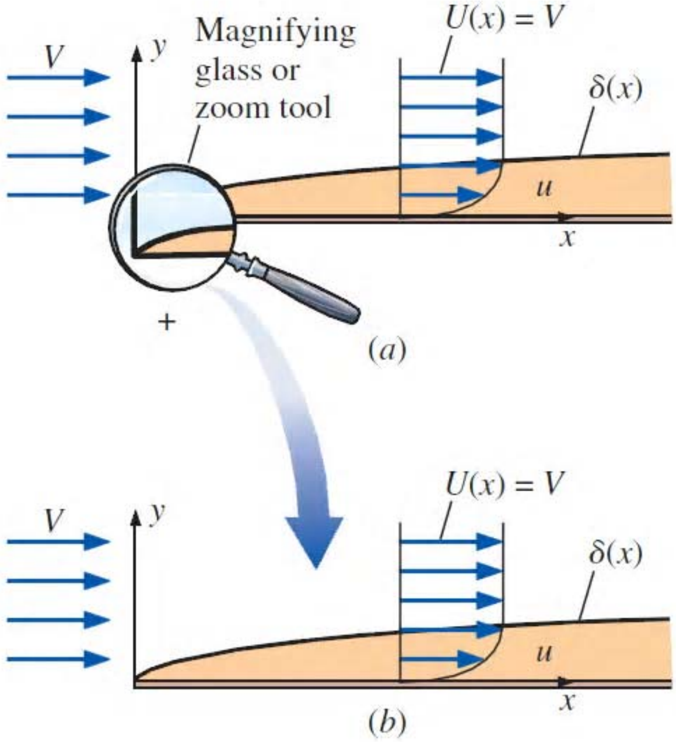


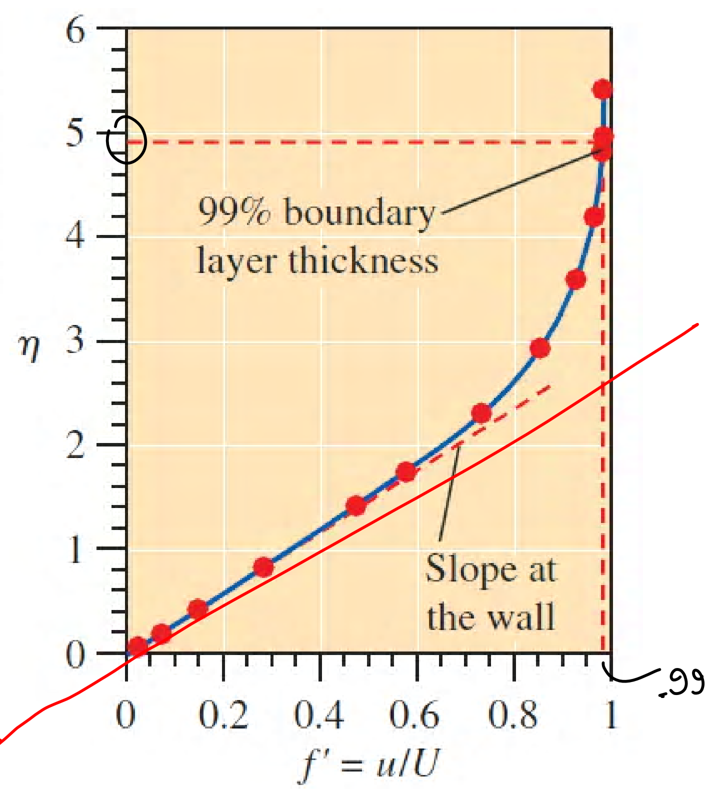
FIGURE 10–98
A useful result of the similarity assumption is that the flow looks the same (is *similar*) regardless of how far we zoom in or out; (*a*) view from a distance, as a person might see, (*b*) close-up view, as an ant might see.

The similarity solution itself is tabulated in Table 10-3, and is plotted in Fig. 10-99.

FIGURE 10–99
The Blasius profile in similarity variables for the boundary layer growing on a semi-infinite flat plate. Experimental data (circles) are at $Re_x = 3.64 \times 10^5$.
From Panton (1996).

Blasius sol'n for a flat plate laminar BL

great success



.99

Slope @ wall $\propto \partial u/\partial y$

This one velocity profile, plotted in nondimensional form as above, applies at *any* x- location in the boundary layer.

Step 4 – Calculate quantities of interest (see text for derivations)

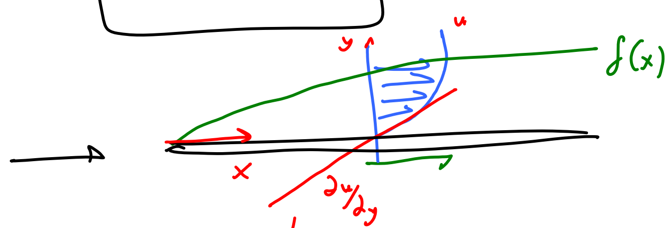
a. δ → 99% BL thickness → y where $u = 0.99\,U$

↓

get from the profile,

$$\boxed{\frac{\delta}{x} = \frac{4.91}{\sqrt{Re_x}}}$$

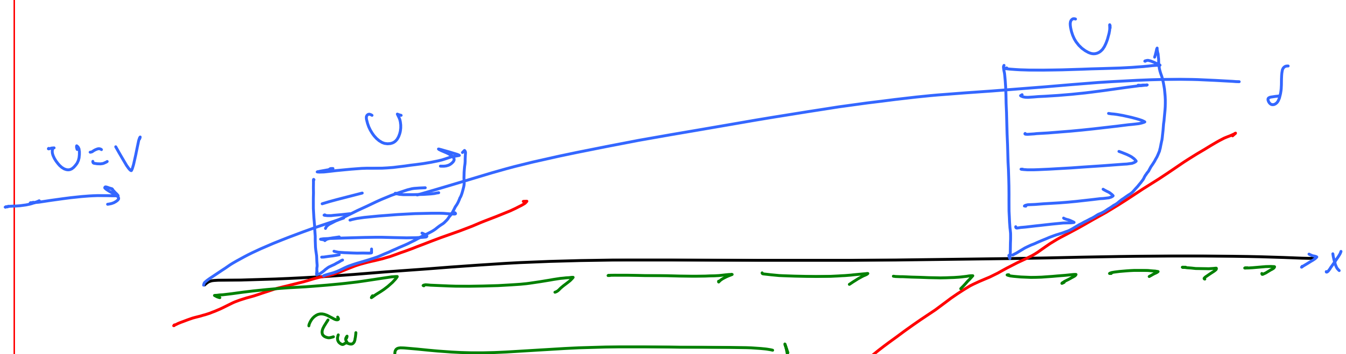
$$\boxed{Re_x = \frac{\rho U x}{\mu} = \frac{U x}{\nu}} \;\star$$



b. τ_w = shear stress @ the wall

$$\boxed{\tau_w = \mu \left.\frac{\partial u}{\partial y}\right)_{y=0}}$$

$$\boxed{C_{f,x} = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.664}{\sqrt{Re_x}}}$$ → local skin-friction coefficient



$\boxed{\tau_w \text{ decreases with } x}$ $\star$

To get total skin friction drag, we have to integrate τ_w from $x=0$ to x

(let b = width of plate into the page)

C. Total skin friction drag

(on one side of the plate) $\rightarrow$

$$F_{D_{friction}} = b\int_0^x \tau_w(x)\,dx$$

$\Big\downarrow$ see text for algebra

Define

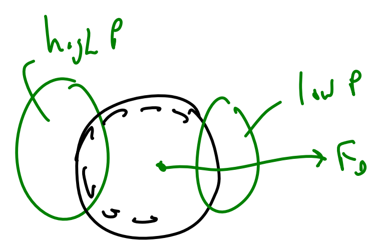
$$\boxed{C_f = \frac{F_{D_{friction}}}{\frac{1}{2}\rho V^2 A} = \frac{1}{x}\int_0^x C_{f,x}\,dx} = \text{Average (or overall) skin-friction coeff.}$$

Like a drag coefficient (recall, $C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$, A = frontal area or planform area)

Here A = planform area = $b \cdot x$

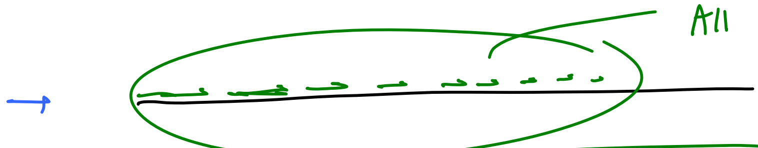
$$\boxed{C_f = \frac{1.33}{\sqrt{Re_x}}} \;\star$$

Two types of drag — skin friction drag & pressure drag



$$F_D = F_{D_{pressure}} + F_{D_{friction}}$$

All we have here is skin friction drag

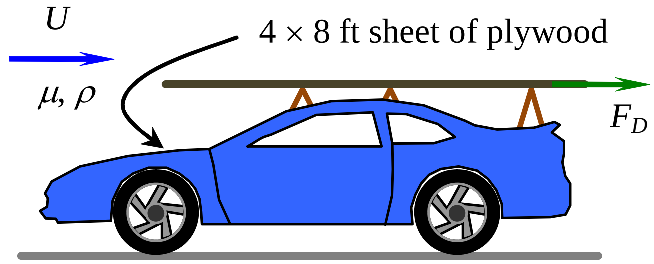


$$C_D = \text{drag coeff} = \frac{F_D}{\frac{1}{2}\rho V^2 \underbrace{A}_{bx}} = C_f$$

For a flat plate

Example: Drag on a sheet of plywood

Given: Craig buys a 4 × 8 ft sheet of plywood at Lowe's and mounts it on the top of his car. He drives (carefully) at 35 mph ≈ 51.3 ft/s. The air density and kinematic viscosity in English units are $\rho = 0.07518$ lbm/ft^3 and $\nu = 1.632\times10^{-4}$ ft^2/s, respectively.



To do: Estimate δ at the end of the plate ($x = L$) and the drag force on the plate.

Solution:

Soln: @ $x = L$, $Re_x = Re_L = \frac{\rho U L}{\mu} = \frac{UL}{\nu}$

$\boxed{\frac{\delta}{x} = \frac{4.91}{\sqrt{Re_x}}}$

$= \frac{(51.3 \text{ ft/s})(8 \text{ ft})}{1.632\times10^{-4} \text{ ft}^2/\text{s}} = \boxed{2.515\times10^6 = Re_L}$

This Re is kind of high to remain laminar

(rule of thumb – by $Re_x = 10^6$, BL becomes turbulent)

Assume BL remains laminar

$\frac{\delta}{x} = \frac{4.91}{\sqrt{Re_x}}$ – w/ $x = L$

$\delta = \frac{4.91}{\sqrt{Re_L}} \cdot L = 0.0248$ ft

$= 0.297$ inches

$\boxed{\delta \approx 0.3 \text{ inches}}$ → indeed – this is very thin!

$F_D = 2 \cdot \frac{1}{\cancel{2}} \rho U^2 C_f A$ → $C_f = \frac{1.33}{\sqrt{Re_L}}$

two sides

4 ft × 8 ft

$F_D = \rho U^2 C_f A$

$= \left(0.07518 \frac{\text{lbm}}{\text{ft}^3}\right)\left(51.3 \frac{\text{ft}}{\text{s}}\right)^2 (0.0008387)(4 \text{ ft})(8 \text{ ft})\left(\frac{1 \text{ lbf}}{32.174 \text{ lbm ft/s}^2}\right) = 0.165$ lbf

$\boxed{F_D \approx 0.17 \text{ lbf}}$ ☆ SMALL !!