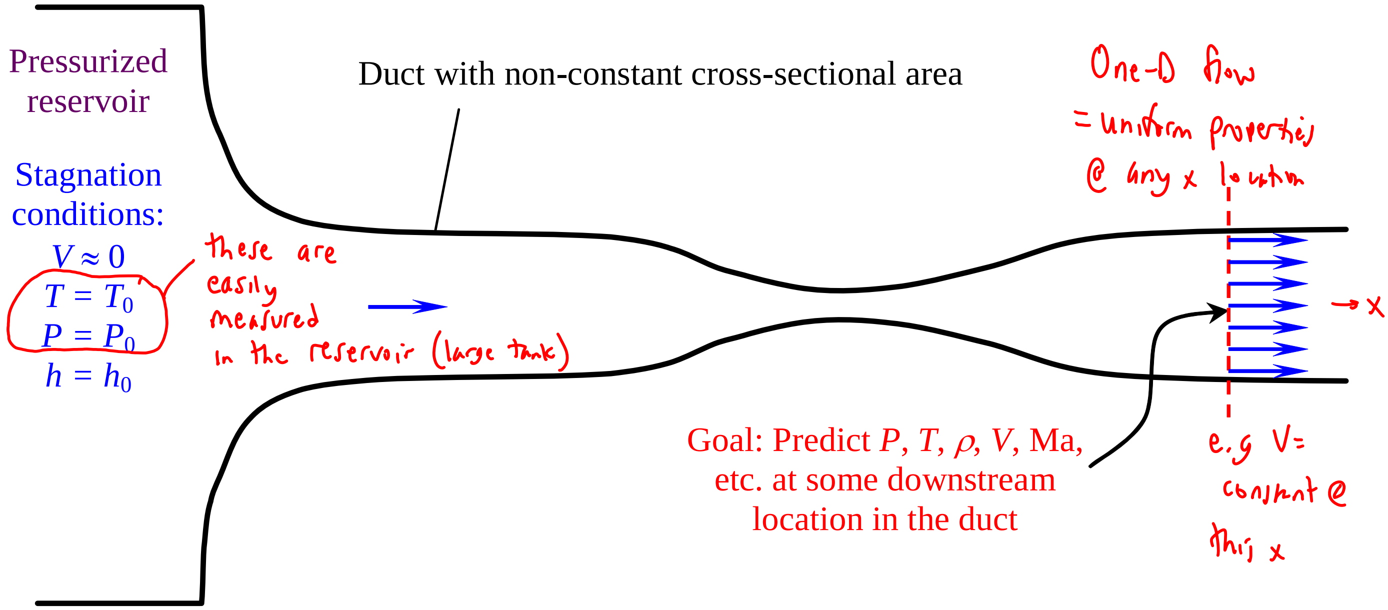
Today, we will:

- Discuss isentropic, compressible, adiabatic flow in ducts
- Discuss converging-diverging ducts, and introduce choking and shock waves

2. One-Dimensional Isentropic Adiabatic Flow in Ducts

a. Setup and equations



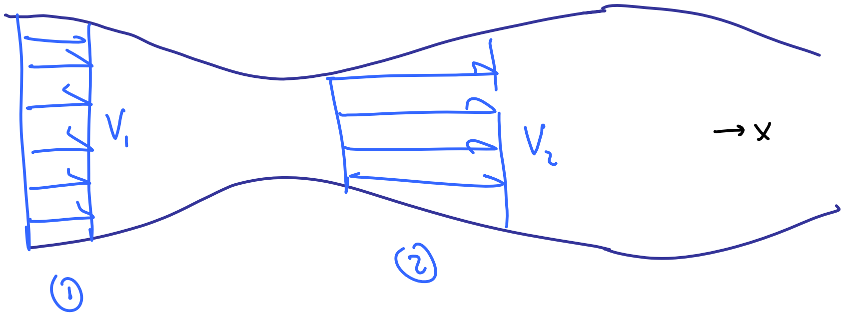
For simplicity, we approximate the flow as ***isentropic*** (negligible friction and other irreversibilities) and ***adiabatic*** (no heat transfer from the air to the surroundings – insulated duct walls).

Equations for isentropic, compressible, adiabatic flow of an ideal gas: from energy eq

- For any ideal gas: $\boxed{\dfrac{T_0}{T} = 1 + \dfrac{k-1}{2}\mathrm{Ma}^2}$ $\boxed{\dfrac{\rho_0}{\rho} = \left(1 + \dfrac{k-1}{2}\mathrm{Ma}^2\right)^{\frac{1}{k-1}}}$ $\boxed{\dfrac{P_0}{P} = \left(1 + \dfrac{k-1}{2}\mathrm{Ma}^2\right)^{\frac{k}{k-1}}}$
- For air (k = 1.4): $\boxed{\dfrac{T_0}{T} = 1 + 0.2\mathrm{Ma}^2}$ $\boxed{\dfrac{\rho_0}{\rho} = \left(1 + 0.2\mathrm{Ma}^2\right)^{2.5}}$ $\boxed{\dfrac{P_0}{P} = \left(1 + 0.2\mathrm{Ma}^2\right)^{3.5}}$
- Or, for air in terms of temperature, $\boxed{\dfrac{\rho_0}{\rho} = \left(\dfrac{T_0}{T}\right)^{2.5}}$ $\boxed{\dfrac{P_0}{P} = \left(\dfrac{T_0}{T}\right)^{3.5}}$ also $\boxed{\dfrac{c_0}{c} = \left(\dfrac{T_0}{T}\right)^{1/2}}$

- Once we know Ma @ some x we can calculate everything else
- How do we calc. Ma at some x? — Ans. We use conservation of mass

b. Isentropic duct flow with area changes



Approx:

1) one-D flow
2) isentropic
3) steady
4) adiabatic
5) ideal gas [air]

Conv. of mass: $\boxed{\dot m = \rho V A}$ @ any x

So far – incompressible flow, $\rho \simeq$ const. $\longrightarrow$ VA = const

see text for derivation: → Cons. of mass

$$\boxed{\frac{dV}{V} = \frac{1}{Ma^2 - 1}\,\frac{dA}{A}}$$

- For $Ma < 1$ (subsonic), $Ma^2 - 1 < 0$

$$\therefore \frac{dV}{V} \propto -\frac{dA}{A} \longrightarrow \boxed{\begin{array}{l} \text{as } A\uparrow\ V\downarrow \\ \text{as } A\downarrow\ V\uparrow \end{array}}$$

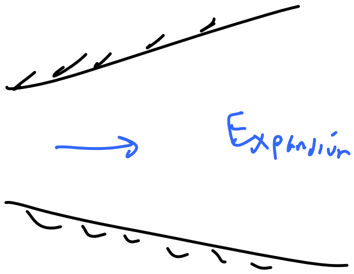
- For $Ma > 1$ (supersonic), $Ma^2 - 1 > 0$

$$\frac{dV}{V} \propto \frac{dA}{A} \longrightarrow \boxed{\begin{array}{l} \text{as } A\uparrow\ V\uparrow \\ \text{as } A\downarrow\ V\downarrow \end{array}}$$ ☆

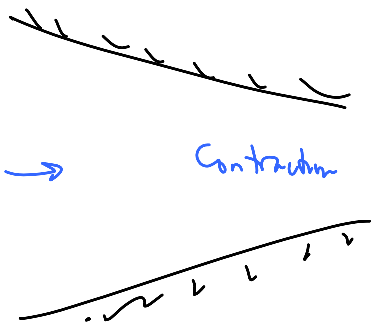
ρ changes rapidly with area change

∴ → we get opposite

Comparison:



	Subsonic	Supersonic
	$A \uparrow$	$A \uparrow$
	$V \downarrow$ $Ma \downarrow$	$V \uparrow$ $Ma \uparrow$
	$P \uparrow$ $\rho \uparrow$	$P \downarrow$ $\rho \downarrow$
	Subsonic diffuser	Supersonic nozzle



	Subsonic	Supersonic
	$A \downarrow$	$A \downarrow$
	$V \uparrow$ $Ma \uparrow$	$V \downarrow$ $Ma \downarrow$
	$P \downarrow$ $\rho \downarrow$	$P \uparrow$ $\rho \uparrow$
	Subsonic nozzle	Supersonic diffuser

★ KNOW THIS

Subsonic water nozzle

$V\uparrow$ as $A\downarrow$

Supersonic rocket nozzle

$V\uparrow$ as $A\uparrow$

We increase the velocity & momentum flux at exit

Accelerating flow in both cases

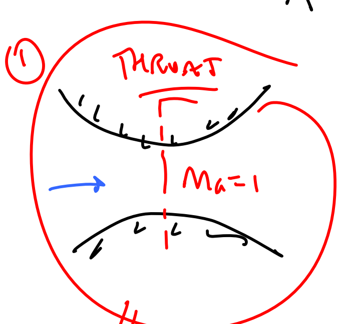
What about $Ma = 1$? (sonic)

Cons. of mass $\rightarrow$ $\dfrac{dA}{A} = (Ma^2 - 1)\dfrac{dV}{V}$

@ $Ma = 1$ $(\ ) = 0$

$$\frac{dA}{A} = 0 \quad \text{when } Ma = 1$$

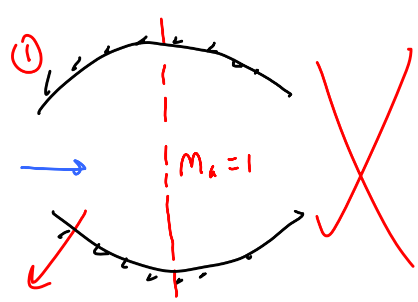
either



This one is possible

If ① is subsonic $Ma < 1$ $\Rightarrow$ $Ma\uparrow$ as $A\downarrow$ ★

or



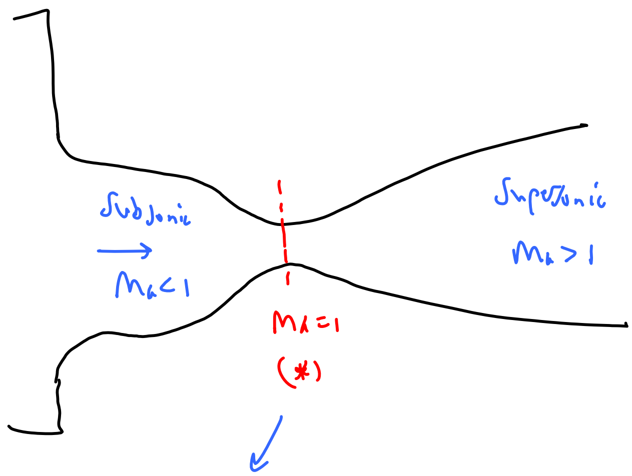
NOT POSSIBLE

If ① is subsonic, $Ma < 1$, but $Ma\downarrow$ as $A\uparrow$

If ① is supersonic, $Ma > 1$, but $Ma\uparrow$ as $A\uparrow$

Conclusion: For 1-D isentropic adiabatic steady duct flow of an ideal gas, sonic conditions (*) can occur only at a throat (minimum area)

To get supersonic flow in a duct, we must have a throat



let A^* be the throat area when the flow is sonic @ throat

can derive eqs for A^* & Ma, T, P, etc. based on the sonic or critical conditions (*)

(1) For air: $\boxed{\dfrac{A}{A^*}=\dfrac{1}{\text{Ma}}\dfrac{\left(1+0.2\text{Ma}^2\right)^3}{1.728}}$ and $\boxed{\dfrac{T_0}{T^*}=\dfrac{k+1}{2}=1.2}$ from which we get

$$\boxed{\frac{\rho_0}{\rho^*}=\left(\frac{T_0}{T^*}\right)^{2.5}=1.2^{2.5}=1.577}\quad\boxed{\frac{P_0}{P^*}=\left(\frac{T_0}{T^*}\right)^{3.5}=1.2^{3.5}=1.893}\quad\boxed{\frac{c_0}{c^*}=\left(\frac{T_0}{T^*}\right)^{1/2}=1.2^{1/2}=1.095}$$

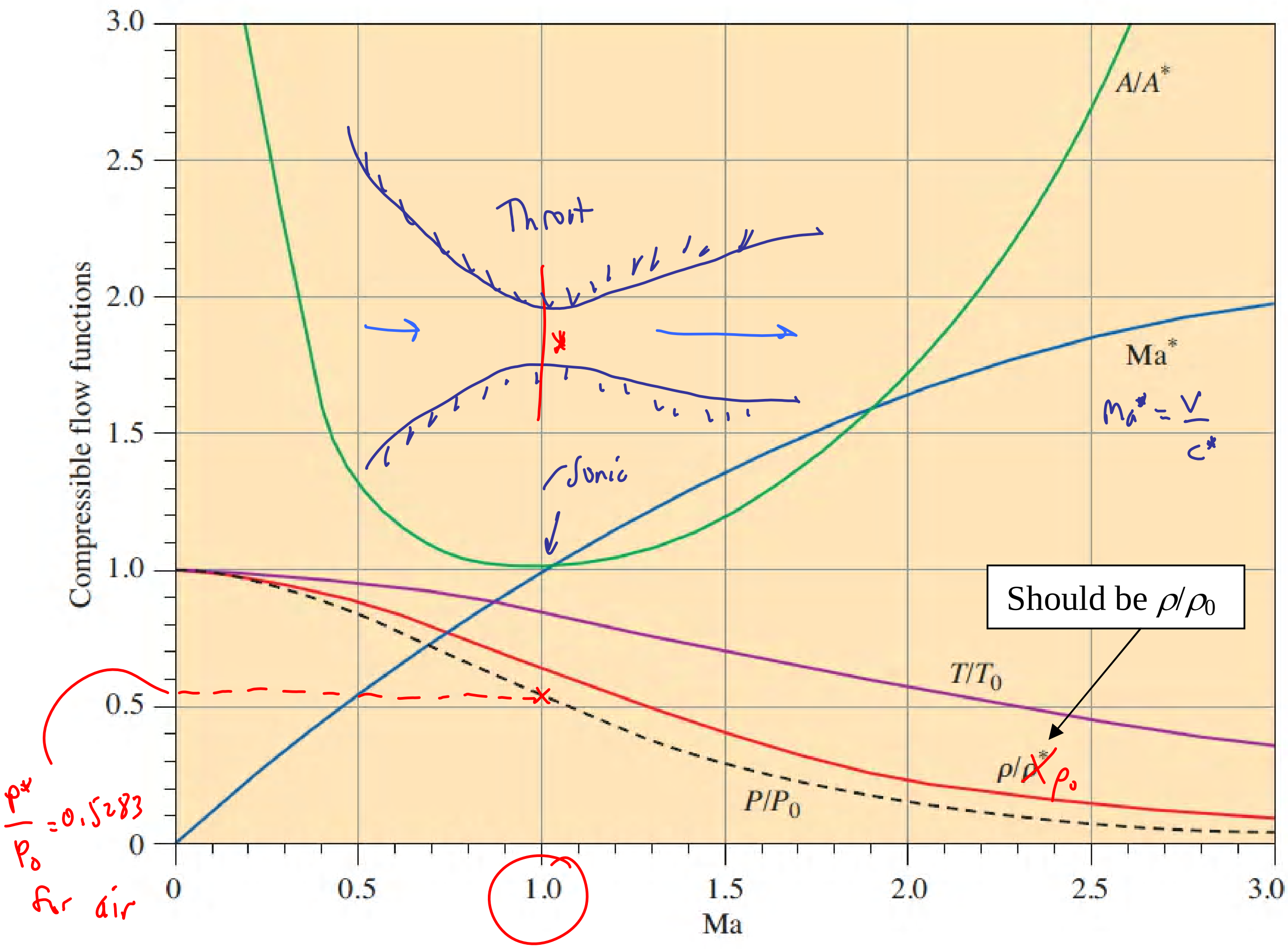
But these are usually listed in terms of their inverses, i.e.,

$$\boxed{\frac{T^*}{T_0}=\frac{2}{k+1}=0.8333}\quad\boxed{\frac{\rho^*}{\rho_0}=0.6339}\quad\boxed{\frac{P^*}{P_0}=0.5283}\quad\boxed{\frac{c^*}{c_0}=0.9129}$$ = **sonic** or **critical** values

We also define a **critical Mach number**, Ma^*, as

$$\boxed{\text{Ma}^*=\frac{V}{c^*}=\frac{V}{c}\frac{c}{c^*}=\frac{\text{Ma}\cdot c}{c^*}=\frac{\text{Ma}\cdot\sqrt{kRT}}{\sqrt{kRT^*}}=\text{Ma}\sqrt{\frac{T}{T^*}}}$$

We plot all these as functions of Mach number in Appendix A-13 (for air):



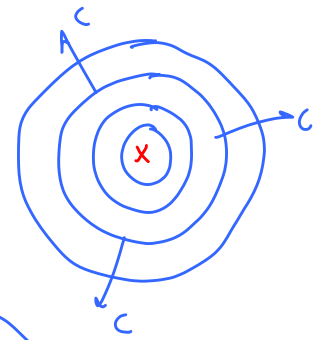
NOTE: Eq. (1) at top of page has two solutions for Ma — one

☆ subsonic (Ma < 1) BEFORE THE THROAT & the other supersonic (Ma > 1) after the throat.

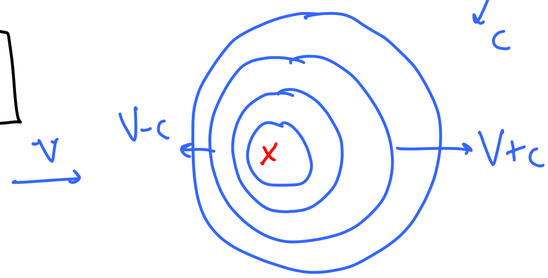
3. Choking

Look at disturbances in pressure, like sound

a) No flow $\boxed{V=0}$

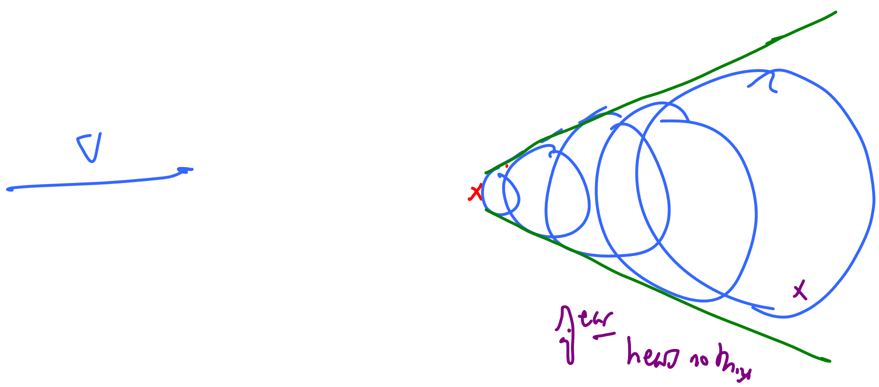


b) $\boxed{V<c}$

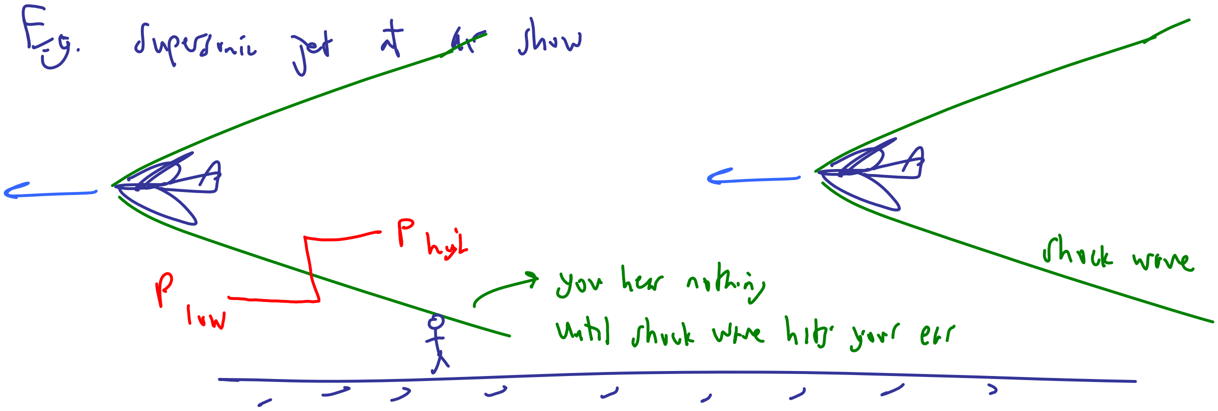


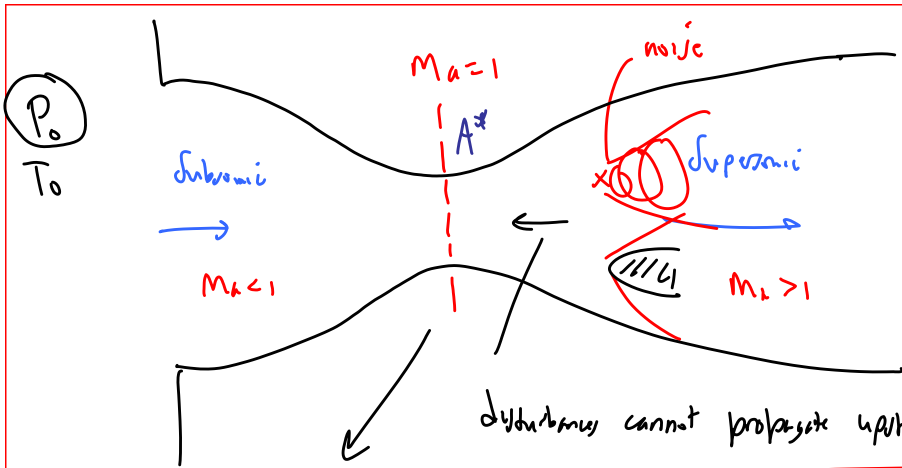
c) $\boxed{V=c}$

d) $\boxed{V>c}$



E.g. supersonic jet at ~~air~~ show





P_b = back pressure

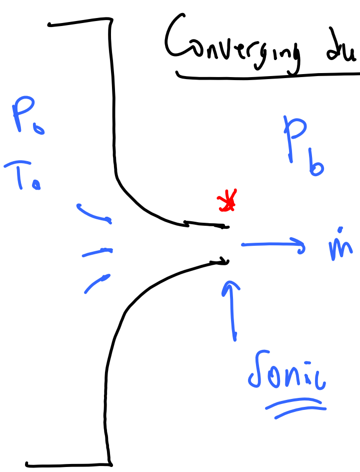
disturbances cannot propagate upstream

★ **Choked condition** — no disturbance or change in downstream flow affects the upstream flow

$\dot{M}$ = mass flow rate reaches a limit at the throat

$$\boxed{\dot{m}_{max} = \frac{0.6847\, P_o A^*}{\sqrt{R T_o}}}$$

Converging duct



$\dot{m}$ = max

If P_b (back pressure) is small enough compared to P_o (reservoir pressure)

[for air, P_b must be less than 0.5283 — see previous plot,

$\frac{P}{P_o} = 0.5283$ when $Ma = 1$

(sonic or critical $*$ condition)]

Then the flow at exit will be sonic & choked

E.g., bicycle tire → $P_o \approx 64$ psia

& $P_b = 14.7$ psia → $\frac{P_b}{P_o} \approx 0.23$

∴ Flow at exit is sonic