Dynamic System Response

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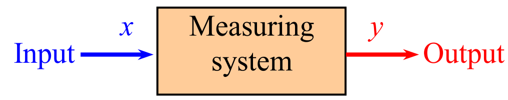
Introduction

- In an *ideal* world, sensors respond instantly to changes in the parameter being measured.
- In the *real* world, however, sensors require some time to adjust to changes, and in many cases exhibit oscillations that take some time to die out.
- In this learning module, we discuss the dynamic system response of sensors and their associated electronic circuits. [Much of this material is also covered in M E 370 – Vibration of Mechanical Systems.]

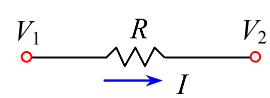
Dynamic systems



- Consider some generic measuring system with an input and an output, as sketched to the right.
- The ***input*** is the physical quantity or property being measured, such as pressure, temperature, velocity, strain, etc. The input is given the symbol x, and is formally called the ***measurand***.
- The ***measuring system*** converts the measurand into something different, so that we can read, record, and/or analyze it. The measuring system may be a ***sensor*** like a strain gage (converts strain directly into a change in resistance) or thermocouple (converts temperature directly into a voltage), or a ***transducer*** like a pressure transducer (converts pressure into a voltage or current).
- The ***output*** may be mechanical or electrical, and its value changes as the measurand changes. The output is given the symbol y.
- The input (measurand) can be either ***static*** (steady within the time of measurement) or ***dynamic*** (unsteady).
- If the measurand is static, the output is generally some factor K times the input, $\boxed{y = Kx}$, where K is called the ***static sensitivity*** of the measuring system.
- For time-dependent (unsteady or dynamic) measurements, the behavior is described by a differential equation. Such systems are called ***dynamic systems***, and their behavior is called ***dynamic system response***.
- In this learning module, only ***linear*** measuring systems are considered. In other words, for static signals, y has a *linear* relationship with x, namely, $y = Kx$, rather than some nonlinear relationship like $y = K_1x + K_2x^2$.

Order of a dynamic system

- Consider the case where measurand x is not constant (static), but is changing with time (dynamic), $x = x(t)$.
- In an ***ideal*** measuring system, output y would respond *instantaneously* to changes in x, $\boxed{y(t) = Kx(t)}$.
- We define n as the ***order of the dynamic system***.
- The order of an ideal dynamic system is zero, i.e., $n = 0$. An ideal measuring system is thus also called a ***zero-order dynamic system***.
- No real system is perfectly ideal, but a simple resistor circuit comes close. Consider the resistor as the system, with voltage drop as the input, and current through the resistor as the output, as sketched to the right.



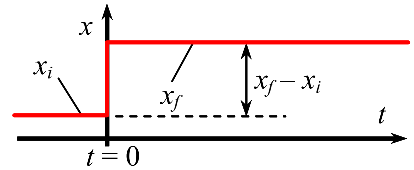
- For an ideal resistor, Ohm's law is satisfied at all times: $\boxed{\Delta V = V_1 - V_2 = IR}$. As the voltage drop changes, the current changes instantaneously to satisfy Ohm's law at any instant in time.
- A real-life resistor has an extremely fast (though not *instantaneous*) time response, and can be approximated as an ideal resistor for most practical applications.
- Thus, a simple resistor circuit is one of the closest examples we have of an ideal system.
- In most real measuring systems, the output does *not* respond instantly to changes in the measurand, and these systems are thus *not* ideal (zero-order) dynamic systems.
- In fact, most real systems behave as either ***first-*** ($n = 1$) or ***second-*** ($n = 2$) order dynamic systems.

Governing differential equation

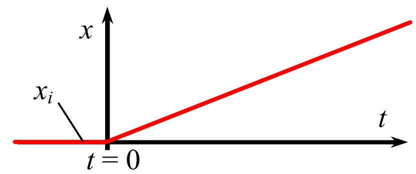
- We express the relationship between input and output for a general measuring system as an ***ordinary differential equation*** (***ODE***) of the form $\boxed{a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = bx}$, where:
  - n is the order of the system (the equation is an n^{th}-order ODE).
  - $x = x(t)$ is the input or measurand, also called the ***forcing function***.
  - $y = y(t)$ is the output.

- a_0, a_1, a_2, ..., a_n are the ***coefficients of the ODE***, assumed in this discussion to be constants.

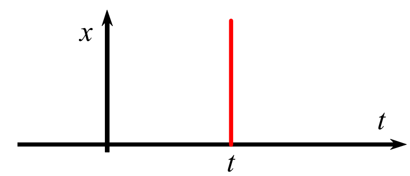
- An n^{th}-order system requires $n + 1$ coefficients (a_0, a_1, a_2, ..., a_n). These coefficients characterize the system.
- When solved, an n^{th}-order system generates n ***constants of integration*** that must be determined by ***boundary conditions***.
- The ***dynamic system response*** of the system is typically tested with one of four types of inputs:
  - ***Step input*** – a sudden change in the measurand at time $t = 0$, as sketched to the right. The step input is used to measure the ***time response*** of the system. x_i and x_f are the initial and final values of x respectively.



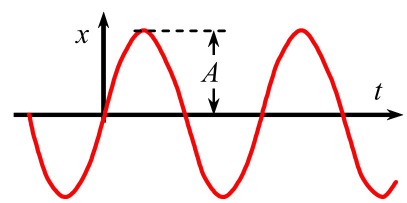
  - ***Ramp input*** – a linear increase in the measurand, starting at time $t = 0$, as sketched to the right. The ramp input provides an alternate way to measure the time response of the system. As shown, the initial value of x is typically zero. In real life, x reaches some upper limit, but we are not concerned here with the final value of x.



  - ***Impulse input*** – a sudden spike in the measurand, at some time t, as sketched to the right. Ideally, the spike has infinite amplitude and zero thickness. In reality, the spike has large amplitude and occurs over a very short interval of time. The impulse input provides another alternate way to measure the time response of the system.



  - ***Sinusoidal input*** – the measurand is a periodic sine wave of frequency f and amplitude A, as sketched to the right. When the frequency of the sine wave is changed, the sinusoidal input provides a way to measure the ***frequency response*** of the system. This input is most useful for testing filters, as discussed in a previous learning module.



- In this learning module, the *step input* is utilized most of the time.

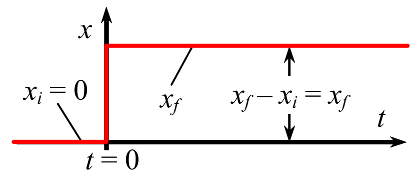
Zero-order measurement system

- **ODE**:
  - Let $n = 0$. The ODE reduces to $a_0 y = bx$.
  - The above equation is re-written as $\boxed{y = Kx}$, where $\boxed{K = b/a_0}$.
  - K is called the ***static sensitivity*** (or simply the ***sensitivity***) of the measuring system.
- **Solution for any input**:
  - This is no longer an ODE since all the derivatives have dropped out.
  - Nothing needs to be solved here, since the solution is already present in the equation itself, $\boxed{y = Kx}$.
  - The solution is independent of time.
  - The output simply follows the input exactly, with some constant K as a multiplicative factor.
- **Physical examples**:
  - No real measurement system is perfectly ideal or zero-order.
  - An electrical circuit with only resistors comes very close to zero-order, as discussed above.

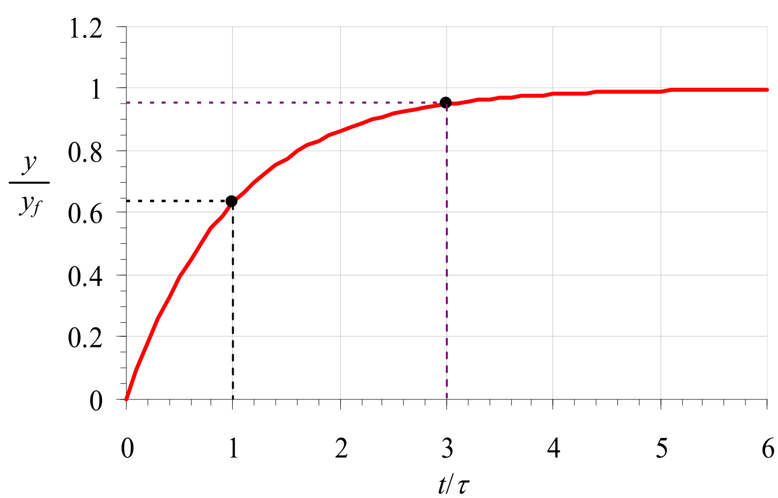
First-order measurement system

- **ODE**:
  - Let order $n = 1$. The ODE reduces to $a_1 \dfrac{dy}{dt} + a_0 y = bx$.
  - Dividing each term by a_0, we get $\dfrac{a_1}{a_0}\dfrac{dy}{dt} + y = \dfrac{b}{a_0}x$ or $\boxed{\tau \dfrac{dy}{dt} + y = Kx}$, where $\boxed{K = b/a_0}$.
  - K is called the ***static sensitivity*** of the measuring system, as defined previously, and a new constant τ called the ***first-order time constant*** has been defined as $\boxed{\tau = a_1/a_0}$.
- **Solution for a step function input**:
  - You may need to dust off your old differential equations book to solve this ODE!

- For ***nonhomogeneous ODEs*** (those with non-zero right hand sides) like the above, the solution is the sum of a ***general*** (homogeneous) part and a ***particular*** (nonhomogeneous) part in which the right hand side takes the actual form of the forcing function, $x(t)$ times K, namely $\boxed{y(t)=y_{\text{general}}(t)+y_{\text{particular}}(t)}$.
- The *general* solution is solved first:
  - The homogeneous part of the differential equation is $\tau\frac{dy}{dt}+y=0$, or $\boxed{\frac{dy}{dt}+\frac{1}{\tau}y=0}$.
  - To solve this homogeneous equation, an ***auxiliary equation*** is formed and solved for the roots m, namely $\left(D+\frac{1}{\tau}\right)y=0$, where $D=\frac{d}{dt}$.
  - This equation has only one root, namely $\boxed{m=-\frac{1}{\tau}}$.
  - Thus, the solution for the general part of the ODE is $y_{\text{general}}=C\exp(mt)$, or $\boxed{y_{\text{general}}=C\,\mathrm{e}^{-t/\tau}}$.
- The *particular* solution is found next:
  - First we plug in the forcing function on the right hand side of the ODE. Here the particular forcing function being applied is that of a step input. We assume that x jumps suddenly from zero (x_i = 0) to some constant value x_f at time $t = 0$ as sketched to the right.
  - The ODE for the particular solution using this forcing function becomes $\boxed{\tau\frac{dy}{dt}+y=Kx_f}$.
  - A simple particular solution is found by inspection, $\boxed{y_{\text{particular}}=Kx_f}$.

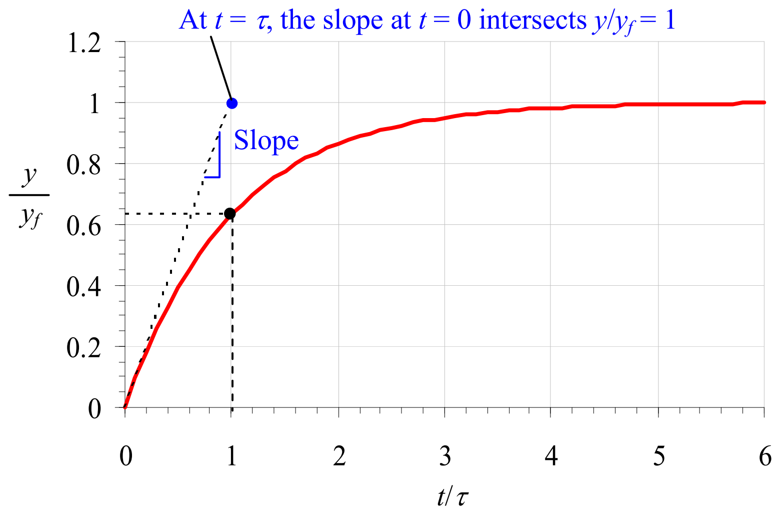


- Finally, then, the solution for $y(t)$ is the sum of the general and particular solutions, $y(t)=y_{\text{general}}(t)+y_{\text{particular}}(t)$, or $\boxed{y(t)=C\,\mathrm{e}^{-t/\tau}+Kx_f}$.
- Notice that there is *one* constant of integration C since this is a first-order system.
- Constant C is found by invoking ***boundary conditions*** (actually ***initial conditions*** here since the dependent variable is *time*).
- The initial condition at $t = 0$ is $y = 0$. Substitution into our solution for $y(t)$ yields $0=C\,\mathrm{e}^{0}+Kx_f$, and solving for C gives $\boxed{C=-Kx_f}$, since $e^0 = 1$.
- So, $y(t)$ for our step-function input becomes $y(t)=-Kx_f\,\mathrm{e}^{\frac{-t}{\tau}}+Kx_f$, or $\boxed{y(t)=Kx_f\left(1-\mathrm{e}^{-t/\tau}\right)}$.
- We let the final value of y (as t approaches infinity) be denoted as y_f. Since $e^{-\infty} = 0$, $\boxed{y_f=Kx_f}$.
- Then the solution for y is written in ***nondimensional form*** as $\boxed{\frac{y}{y_f}=1-\mathrm{e}^{-t/\tau}}$.

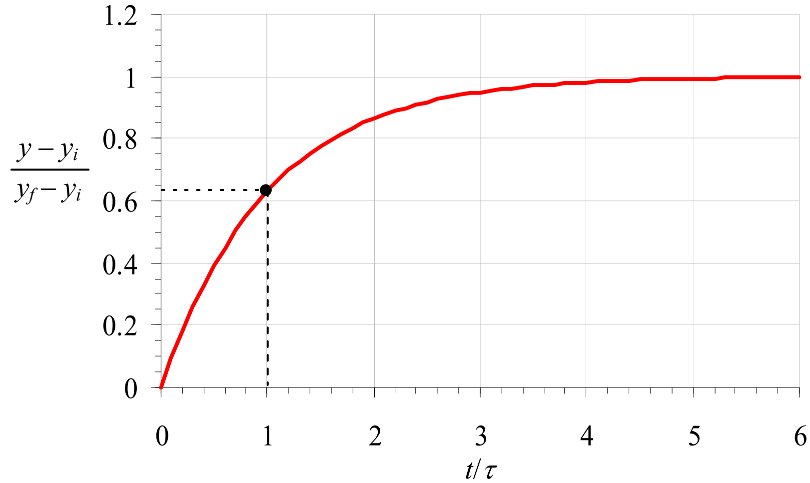


- A plot of nondimensional output y/y_f is shown to the right as a function of nondimensional time t/τ.
- Marked on the plot is the value of y/y_f when t = exactly *one* time constant. At $t = \tau$ ($t/\tau = 1$), $y/y_f=1-\mathrm{e}^{-1}=0.63212...$, or y/y_f ≈ 63.2% after one time constant has elapsed. In other words, after one time constant, y has increased to approximately 63.2% of its final value.

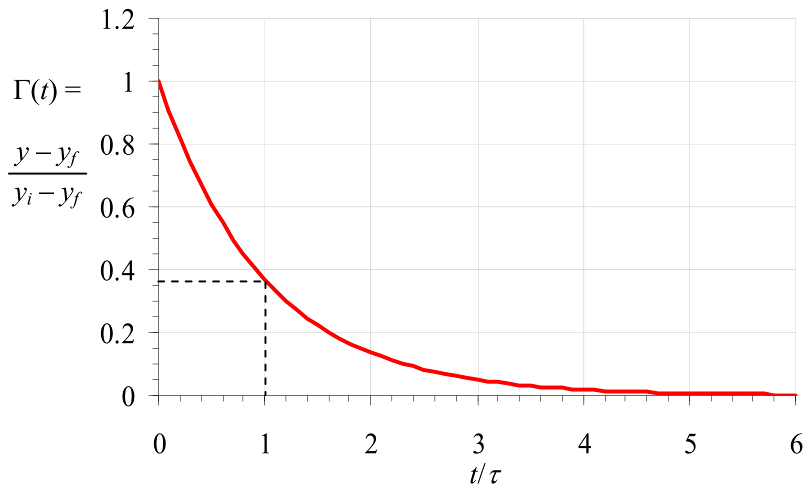
- Also marked on the plot is the value of y/y_f when t is exactly constants. At $t = 3\tau$ ($t/\tau = 3$), $y/y_f = 1 - \mathrm{e}^{-3} = 0.95021...$, or $y/y_f \approx 95.0\%$ after 3 time constants have elapsed. In other words, after three time constants ***(the 95% rise time)***, y has increased to approximately 95.0% of its final value.
- Similarly, after *five* time constants, y is more than 99.3% of y_f. Theoretically it takes infinite time for y to equal y_f, but in practical terms, it is often stated that y "reaches" y_f after about five time constants.
- Shown here are two methods to determine the time constant:
  - The time constant is equal to the time where $y/y_f = 1 - e^{-1} =$ 0.63212..., as discussed above.
  - The time constant is equal to the time where the slope of the output curve at $t = 0$ intercepts $y/y_f = 1$, as plotted to the right.



- The more general case is if $x = x_i$ (some initial value not equal to zero) until $t = 0$, when it jumps suddenly to $x = x_f$ (final value). The corresponding initial and final values of y are y_i and y_f respectively.
- We define a *nondimensional* form of the output. You can show that in such a case, $\boxed{\dfrac{y - y_i}{y_f - y_i} = 1 - \mathrm{e}^{-t/\tau}}$.
- A plot of this nondimensional function looks the same as the above, but the vertical axis changes from simply y/y_f to $(y - y_i)/(y_f - y_i)$, as plotted to the right.

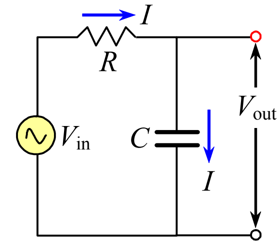


- As indicated on the plot, when $t/\tau = 1$, the nondimensional function $(y - y_i)/(y_f - y_i) =$ 0.63212....
- It is also common to define the ***error fraction*** of the output as $\boxed{\Gamma(t) = \dfrac{y - y_f}{y_i - y_f} = \mathrm{e}^{-t/\tau}}$.
- A plot of the error fraction is shown to the right. As can be seen, the error fraction starts at unity at $t = 0$, and gradually settles to zero for large time.
- At a nondimensional time of 1 ($t/\tau = 1$), $\Gamma(t) = 1 - 0.63212... =$ 0.36788..., as indicated on the plot.



- **Physical examples**:
  - Most temperature measuring devices (thermometers, thermocouples) are first-order measuring systems.
  - An electrical circuit with only resistors and capacitors is a first-order system. For example, simple passive RC low-pass and high-pass filters act as first-order systems.

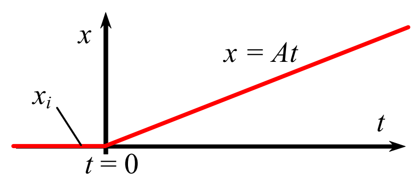
- **Example**:
  Given: A simple passive RC low-pass filter, as sketched to the right.
  To do: Prove that this circuit operates as a first-order system.
  Solution:



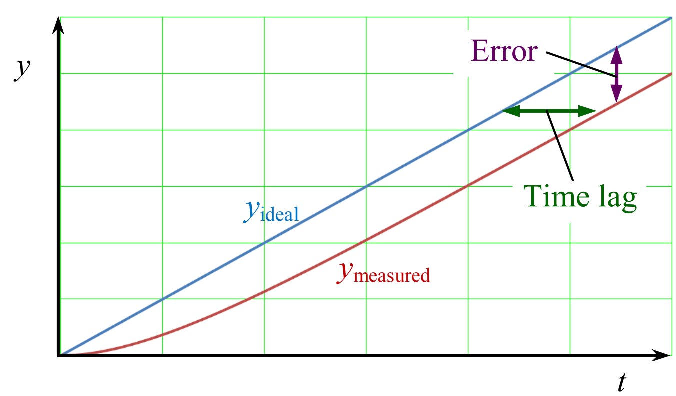
  - We assume that the measuring device has infinite input impedance, so that no current flows through the V_{out} leads. Thus, the current flowing through the resistor must equal the current flowing across the capacitor.
  - Ohm's law provides the current through the resistor, $I = (V_{in} - V_{out}) / R$.
  - The equation for an ideal capacitor provides the current across the capacitor, $I = C\frac{dV_{out}}{dt}$.
  - Equating the current through these two components yields a differential equation for output voltage V_{out}, namely $C\frac{dV_{out}}{dt} = \frac{V_{in} - V_{out}}{R}$, which we rewrite as $\boxed{C\frac{dV_{out}}{dt} + \frac{1}{R}V_{out} = \frac{V_{in}}{R}}$.
  - Since the above differential equation is a first-order ODE, the low-pass filter is indeed a first-order system.

  Discussion: Now that we have the differential equation in standard form, the time constant for this circuit can be obtained. This is left as an exercise.

- **Solution for a ramp input**:
  - Now consider a different input function, namely a ***ramp function*** as shown to the right. Input x is zero until time $t = 0$ at which point it ramps up *linearly* as follows: $\boxed{x = At}$, where A is the ***ramp rate***.

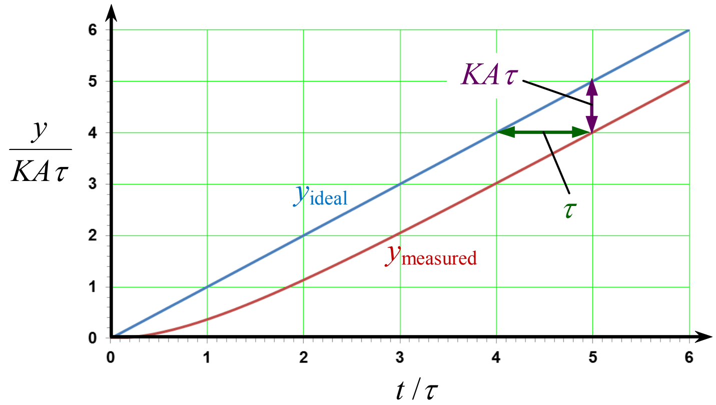


  - A physical example is a container of water that is heated from 20°C to 100°C linearly by supplying constant uniform heat to the water.
  - The order of the system is still $n = 1$. The ODE is the same as previously, namely, $a_1\frac{dy}{dt} + a_0 y = bx$, and we define the first-order time constant in the same way as previously, namely, $\boxed{\tau = a_1 / a_0}$ so that $\boxed{\tau\frac{dy}{dt} + y = Kx}$, where $\boxed{K = b / a_0}$.
  - A zero-order or ideal system would respond instantaneously as input x increases, and the ideal response would be $\boxed{y_{ideal} = Kx = KAt}$. [Another way to think about this: a zero-order ideal system has a first-order time constant of zero ($\tau = 0$), and the system response has *no* delay or lag.]
  - For the *actual* or *measured* first-order system, however, τ is *not* zero, and we need to solve the differential equation. After some algebra, we obtain the solution, $\boxed{y_{measured} = KA\left[t - \tau\left(1 - e^{-(t/\tau)}\right)\right]}$.



  - The ideal and measured responses are compared in the plot to the right. Because the first-order (measured) system does not respond instantaneously, we see a *lag* in the growth of the output or response, $y_{measured}$, compared to the linear growth of y_{ideal}.
  - We notice, however, that at large values of t, the two curves become *parallel* to each other, approaching a constant ***time lag*** (horizontal difference between $y_{measured}$ and y_{ideal}) and a constant ***error*** (vertical difference between $y_{measured}$ and y_{ideal}).

- Equations for the time lag and error are easily obtained by letting $t \to \infty$ in the equation for y_{measured}, which simplifies to $\boxed{y_{\text{measured}} \to KA(t-\tau) \text{ as } t \to \infty}$.
- Comparing to the ideal case, we see that $\boxed{\text{time lag} = \tau}$ and $\boxed{\text{error} = y_{\text{measured}} - y_{\text{ideal}} = -KA\tau}$.
- These values are most clearly visualized by normalizing the above plot. We normalize t by time constant τ and y by the constant $KA\tau$ (which has the same dimensions as y). The normalized plot is shown to the right.



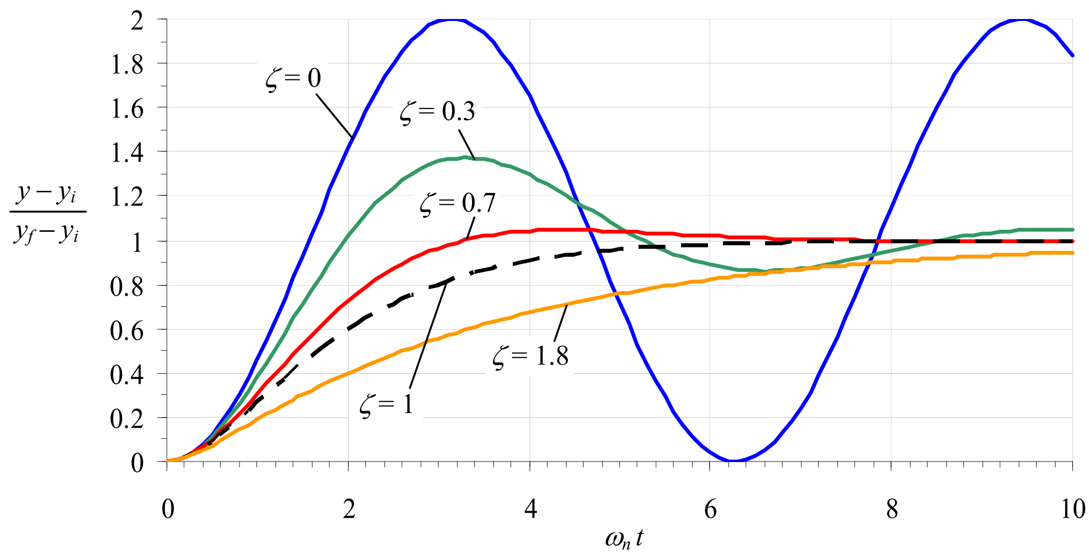
- In practice, this provides an alternative method for measuring the first-order time constant τ. We have two choices:
  - Measure the time lag, which is equal to τ.
  - Measure the *slope*, which is equal to KA, and the *error*, whose magnitude is $KA\tau$, and then calculate τ as $\boxed{\tau = \frac{|\text{error}|}{\text{slope}} = \frac{KA\tau}{KA}}$.

Second-order measurement system

- **ODE**:
  - We set order $n = 2$ for a second-order system. The ODE reduces to $a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = bx$.
  - Dividing each term by a_0, we get $\frac{a_2}{a_0}\frac{d^2 y}{dt^2} + \frac{a_1}{a_0}\frac{dy}{dt} + y = \frac{b}{a_0}x$, or $\boxed{\frac{1}{\omega_n^{\,2}}\frac{d^2 y}{dt^2} + \frac{2\varsigma}{\omega_n}\frac{dy}{dt} + y = Kx}$, where $\boxed{K = b/a_0}$.
  - K is again called the ***static sensitivity*** of the measuring system, as defined previously.
  - Two new constants have been introduced, defined as follows:
    - The ***undamped natural frequency*** is defined as $\boxed{\omega_n = \sqrt{\frac{a_0}{a_2}}}$, and ω_n represents the angular frequency at which the system would oscillate if there were no damping.
    - The ***damping ratio*** (identified by the Greek letter ***zeta***) is defined as $\boxed{\varsigma = \frac{a_1}{2\sqrt{a_0 a_2}}}$.
  - Whereas a first-order system requires only *one* parameter (the first-order time constant τ) to describe its time response, a second-order system requires *two* parameters to describe its time response, namely the undamped natural frequency ω_n and the damping ratio ζ.
- **Solution for a step function input**:
  - As with first-order systems, the solution for second-order systems is the sum of a ***general*** (homogeneous) part and a ***particular*** (nonhomogeneous) part in which the right hand side takes the actual form of the forcing function, $Kx(t)$.
  - Details of the solution are not included here; rather a summary of the results is given.
  - Two constants of integration are produced, since this is a second-order system. The constants are found by invoking initial conditions, as was done above for the first-order system.
  - The solution below is for the step-input case where $x = x_i$ (initial value) until $t = 0$, when it jumps suddenly to $x = x_f$ (final value). The corresponding initial and final values of y are y_i and y_f respectively.
  - Final response y_f is also called the ***equilibrium response***, defined as the value of y_f as $t \to \infty$.
  - For second-order systems, there is not really a simple time constant, but the undamped natural frequency ω_n is used to nondimensionalize time. The ***nondimensional time*** is defined as $\boxed{\omega_n t}$.
  - The solution for a step function input is summarized below. There are three equations, depending on whether the damping ratio ζ is less than unity, equal to unity, or greater than unity:

- For $\zeta < 1$, the system is ***underdamped***, and the nondimensional output is given by $\boxed{\dfrac{y-y_i}{y_f-y_i}=1-\mathrm{e}^{-\zeta\omega_n t}\left[\dfrac{1}{\sqrt{1-\zeta^2}}\sin\left(\omega_n t\sqrt{1-\zeta^2}+\phi\right)\right]}$, where ***phase shift*** ϕ is $\boxed{\phi=\sin^{-1}\left(\sqrt{1-\zeta^2}\right)}$.
- For $\zeta = 1$, the system is ***critically damped***, and the nondimensional output is given by $\boxed{\dfrac{y-y_i}{y_f-y_i}=1-\mathrm{e}^{-\omega_n t}\left(1+\omega_n t\right)}$, and there is no phase shift.
- For $\zeta > 1$, the system is ***overdamped***, and the nondimensional output is given by $\boxed{\dfrac{y-y_i}{y_f-y_i}=1-\mathrm{e}^{-\zeta\omega_n t}\left[\cosh\left(\omega_n t\sqrt{\zeta^2-1}\right)+\dfrac{\zeta}{\sqrt{\zeta^2-1}}\sinh\left(\omega_n t\sqrt{\zeta^2-1}\right)\right]}$.

- Nondimensional output is plotted as a function of nondimensional time for several values of damping ratio below.



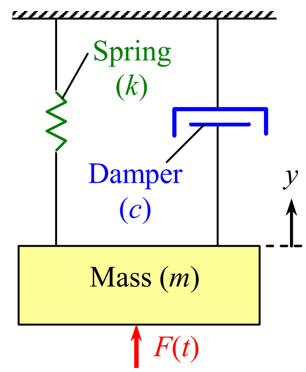
- When the damping ratio is zero (**blue curve**), the system oscillates around the final value forever, never reaching the final value. This is the ***undamped*** case. Physically, there is nothing to damp out the fluctuations (coefficient a_1 in the original differential equation is zero). Notice that the output oscillates with a period slightly greater than 6 in nondimensional time units. This is because the oscillation occurs at *angular frequency* ω_n (in radians/s), which is a factor of 2π greater than the actual *physical frequency* f_n (in Hz). The ***undamped natural physical frequency*** f_n (in Hz) is given by $\boxed{f_n=\dfrac{\omega_n}{2\pi}=\dfrac{1}{2\pi}\sqrt{\dfrac{a_0}{a_2}}}$.
- When the damping ratio is non-zero, but less than one (**green curve**), the system overshoots the final value, and then oscillates around the final value, eventually converging. This is called ***underdamped***. Physically, there is some damping, but not enough to eliminate the oscillations. As seen on the plot, the period of oscillation for the underdamped case is somewhat *larger* than that of the undamped case. This means that, inversely, the oscillation frequency ω_d, which is called the ***damped natural frequency*** (subscript "*d*" stands for "damped"), is somewhat *smaller* than the undamped natural frequency. Some authors call ω_d the ***ringing frequency*** (more properly ***angular ringing frequency***). Mathematically, it can be shown that $\boxed{\omega_d=\omega_n\sqrt{1-\zeta^2}}$. In terms of physical frequency, $\boxed{f_d=f_n\sqrt{1-\zeta^2}}$.
- When the damping ratio is exactly one (the **black dashed curve**), the output has no overshoot and no oscillation. This is called ***critically damped***. Physically, the damping is just enough to eliminate oscillations. We see from the above equation that the ringing frequency of a critically damped system is zero, or inversely, the period of oscillation is infinity – there is no oscillation.
- When the damping ratio is greater than one (**orange curve**), the output has no overshoot, and asymptotes to the final value without oscillation, but more slowly than does the critically damped case. This is called ***overdamped***. Physically, there is *too much damping*. The ringing frequency is non-existent or undefined.

An overdamped second-order system behaves somewhat like a first-order system in that there is no overshoot. However, the shapes of the two curves are not identical. In particular, the behavior at time zero is much different – namely, the slope at $t = 0$ is non-zero for a first-order system, but zero for a second-order system.

- Finally, one other case is plotted on the graph. Namely one in which the damping ratio is 0.7 (**red curve**). This case is called ***optimally damped***. Physically, although there is not enough damping to eliminate oscillations, the output settles down to the final value very quickly. In fact, at optimal damping, the output settles more quickly than for any other damping ratio. The maximum overshoot occurs at a nondimensional time of approximately 4.4, and the magnitude of the first overshoot is only about 4.6% of the magnitude of $(y_f - y_i)$. Subsequent undershoots and overshoots are significantly smaller than this, and the oscillation dies out very quickly.
- In practice, if overshoots cannot be tolerated, the parameters should be adjusted as necessary to make the damping ratio = 1 (critically damped) or > 1 (overdamped). If the fastest response time is desired, and a small overshoot can be tolerated, the damping ratio should be around 0.7 (optimally damped).

- **Physical examples**:
  - A system consisting of a spring, a mass, and a damper can model many physical devices, such as components of an automobile, vibrating beams, etc.
  - An electrical circuit with only resistors, capacitors, and inductors acts as a second-order system. For example, a simple passive LRC low-pass filter acts as a second-order system.

- **Example**: ***Second-order spring-mass-damper system***

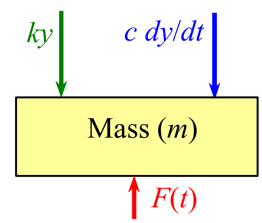


Given: The most common example of a second-order system is that of a spring, mass, and damper. An idealized schematic is sketched to the right. The spring has ***spring constant*** k, and the damper has ***damping constant*** c (also sometimes called the ***damping coefficient***, and sometimes the Greek letter lambda λ is used instead of c). The ***mass*** is m, and the ***forcing function*** is some force $F(t)$. ***Output*** y is defined as the vertical position of the top of the mass. The gravity force may act in any direction, but is not considered as a separate force here; the effect of gravity is included in the forcing function.

To do: Prove that this system operates as a second-order system.

Solution:

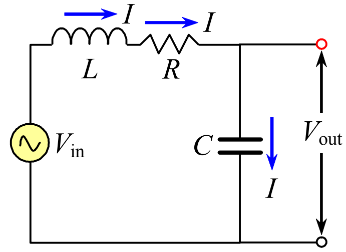
- To generate a differential equation for position y as a function of time, a free body diagram of the mass is drawn to the right.



- We apply Newton's second law, $\sum F = ma$, or $m\frac{d^2y}{dt^2} = \sum F$, and sum all the forces acting on the mass: $m\frac{d^2y}{dt^2} = -c\frac{dy}{dt} - ky + F(t)$, which we write in more standard form as $\boxed{m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F(t)}$.
- This ODE is in the same form as the general ODE for a second-order system. Therefore, a spring-mass-damper system responds as a second-order system.
- Equating this ODE with the standard 2nd-order ODE, we calculate $\boxed{\omega_n = \sqrt{\frac{k}{m}}}$, $\boxed{\zeta = \frac{c}{2\sqrt{km}}}$, and $\boxed{K = \frac{1}{k}}$ (don't confuse lower case k and upper case K). When $\zeta < 1$, the actual frequency is $\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$.
- At rest, the weight of the mass is the only force that acts. Therefore the forcing function, $F(t)$ is simply a constant equal to $-mg$. This can be thought of as the *final* value of the forcing function, x_f, since any oscillation of the system will eventually die out to this state of rest. The corresponding final output (displacement) is y_f (*Note*: Here, y_f is actually a *negative* value because of our definition of y).
- If the mass is displaced upward and held still, this represents some *initial* value of the forcing function, with corresponding displacement y_i.

Discussion: If the mass is suddenly released from its initial displacement location y_i, it will eventually fall back down to its final displacement location y_f. Since this scenario is identical to the step function forcing, the behavior of displacement y behaves precisely as described above, i.e., as a second-order system, with or without oscillations, depending on the damping ratio.

- **Example**: ***Second-order low-pass filter system***
  Given: A simple passive LRC low-pass filter, as sketched to the right.
  To do: Prove that this filter operates as a second-order dynamic system.
  Solution:



  - We assume that the measuring device has infinite input impedance, so that no current flows through the V_{out} leads. Thus, at any instant in time, the current flowing through the resistor must equal the current flowing through the inductor, which must in turn equal the current flowing across the capacitor.
  - The voltage drop across the inductor, resistor, and capacitor must add up to the input voltage V_{in} at any instant in time, namely, $V_{\text{in}} = V_L + V_R + V_C$, where V_C is the output voltage, V_{out} since we are measuring output voltage across the capacitor.
  - Differentiating, $\boxed{\frac{dV_{\text{in}}}{dt} = \frac{dV_R}{dt} + \frac{dV_L}{dt} + \frac{dV_C}{dt}}$.
  - Ohm's law provides an equation for the rate of change of voltage across the resistor, $V_R = IR$, which we differentiate to get $\boxed{\frac{dV_R}{dt} = R\frac{dI}{dt}}$.
  - The ideal equation for an inductor provides an equation for the rate of change of voltage across the inductor, $V_L = L\frac{dI}{dt}$, which we rewrite as $\boxed{\frac{dV_L}{dt} = L\frac{d^2I}{dt^2}}$.
  - The ideal equation for a capacitor provides an equation for the rate of change of voltage across the capacitor, $I = C\frac{dV_C}{dt}$, which we rewrite as $\boxed{\frac{dV_C}{dt} = \frac{I}{C}}$.
  - Finally, substitution of the above three equations into the equation for the derivative of V_{in} yields $\frac{dV_{\text{in}}}{dt} = \frac{dV_R}{dt} + \frac{dV_L}{dt} + \frac{dV_C}{dt} = R\frac{dI}{dt} + L\frac{d^2I}{dt^2} + \frac{I}{C}$, or $\boxed{L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = \frac{dV_{\text{in}}}{dt}}$.
  - Since the above differential equation is a second-order ODE, the LRC low-pass filter is indeed a second-order system.

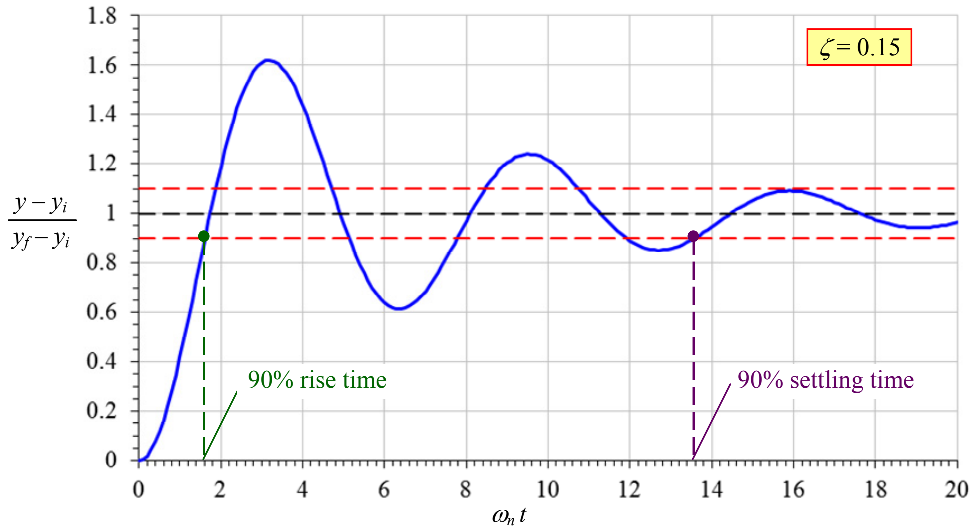
  Discussion: The output variable in this case is current *I* rather than output voltage V_{out}, but nevertheless, we have shown that this system is a second-order dynamic system.

- **Example**:
  Given: The LRC low-pass filter of the previous example.
  To do: Calculate the undamped natural frequency and the damping ratio of this low-pass filter.
  Solution:
  - The differential equation of the previous example is already arranged in the same form as the general ODE for a second-order system. Thus the coefficients can be determined by inspection. The coefficients are $\boxed{a_0 = \frac{1}{C}}$, $\boxed{a_1 = R}$, and $\boxed{a_2 = L}$.
  - The undamped natural frequency and damping ratio of this system are then easily calculated from their definitions, i.e., $\boxed{\omega_n = \sqrt{\frac{a_0}{a_2}} = \sqrt{\frac{1}{LC}}}$ and $\boxed{\zeta = \frac{a_1}{2\sqrt{a_0 a_2}} = \frac{R}{2\sqrt{L/C}} = \frac{R}{2}\sqrt{\frac{C}{L}}}$.

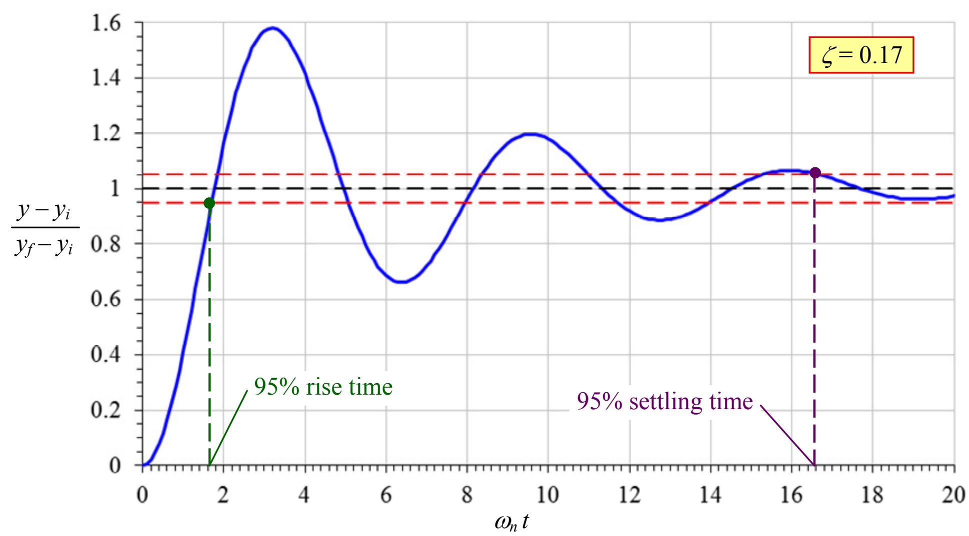
  Discussion: For given values of *L*, *R*, and *C*, we can calculate the damping ratio to determine whether the system would (or would not) have overshoot if a sudden change in voltage were to be applied.

- **Response times**:
  - A first-order dynamic system is characterized by a single parameter, the first-order time constant τ.
  - A second-order dynamic system is characterized by *two* parameters, which we defined above as the undamped natural frequency ω_n and the damping ratio ζ.
  - In terms of time constants, a first-order dynamic system is characterized by a single time constant τ.
  - To characterize a second-order dynamic system in terms of time constants, *two* time constants are required. For underdamped second-order systems, these are defined as the ***rise time*** and the ***settling time***.

- First, we choose the desired ***level*** of the rise time and settling time, namely, how close we want to get to the final nondimensional value. Typical levels are 90%, 95%, or 99%, but any level can be chosen.
- For illustration, we choose 90%. Thus,
  - The ***90% rise time*** is defined as the time it takes for the nondimensional output to *first* reach 90% of the final nondimensional output.
  - the ***90% settling time*** (also called the ***90% response time***) is defined as the time it takes for the nondimensional output to *settle* within +/-10% of the final nondimensional output.
- As an example, we pick an underdamped second-order dynamic system with damping ratio $\zeta = 0.15$. The nondimensional output is plotted here versus nondimensional time.

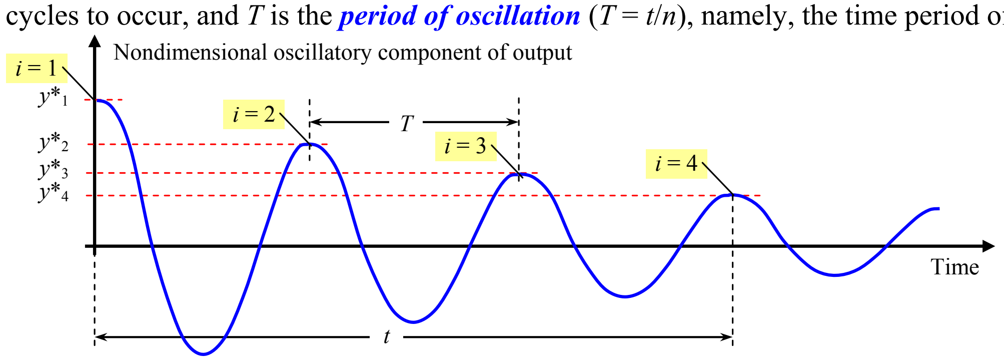


- Also plotted are two dashed red lines at $(y-y_i)/(y_f-y_i) = 0.9$ and 1.1, representing 90% of the final value (10% *below* the final value) and 110% of the final value (10% *above* the final value), respectively.
- The 90% rise time is defined as the *first* time the output crosses the lower dashed red line, i.e., when the nondimensional output reaches a value of 0.90 for the *first* time. ***In this example, the* 90% *rise time is approximately* 1.6 *nondimensional time units***, as indicated on the plot.
- The 90% settling time is defined as the *last* time the output crosses either the lower or upper dashed red line, i.e., when the nondimensional output settles within a value that is +/-10% of the final value. ***In this example, the* 90% *settling time is approximately* 13.6 *nondimensional time units***, as indicated.
- The 90% settling time may occur when the output crosses from *below*, as in this example, or from *above* – we must look for the time when the output crosses *either* of the dashed red lines for the *final* time.
- We may choose a level other than 90%. For example, the case below shows a system with $\zeta = 0.17$, and a level of 95% for our definitions of rise time and settling time, which are indicated on the plot.



- In this case, the output crosses the *upper* limit for the last time (settles from *above*).
- In this case, the two dashed red lines are at nondimensional ouput values of 0.95 and 1.05, representing 95% of the final value (5% *below* the final value) and 105% of the final value (5% *above* the final value), respectively, for +/-5% settling.
- Finally, for an *overdamped* second-order dynamic system, the rise time and settling time are *equal*, because once the nondimensional output crosses the lower red dashed line, it slowly asymptotes to 1, never again crossing either dashed red line – it has settled once it has crossed the lower line.

- **<u>How to estimate the damping ratio</u>**:
  - In the analysis here, we consider only *underdamped* systems ($\zeta < 1$), in which the equation for nondimensional output is $\boxed{y_{\text{norm}} = \dfrac{y - y_i}{y_f - y_i} = 1 - \mathrm{e}^{-\zeta\omega_n t}\left[\dfrac{1}{\sqrt{1-\zeta^2}}\sin\left(\omega_n t\sqrt{1-\zeta^2} + \sin^{-1}\left(\sqrt{1-\zeta^2}\right)\right)\right]}$, coupled with a second equation that relates ω_n and ω_d, namely, $\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$.
  - When the undamped natural frequency ω_n and the damping ratio ζ are *known*, the above equation for nondimensional output $(y - y_i)/(y_f - y_i)$ as a function of time can be used directly (*explicitly*) to predict the behavior of the second-order dynamic system.
  - However, in a typical experiment, we do *not* know ω_n and ζ. Rather, we wish to *solve* for ω_n and ζ. Specifically, we measure the output as a function of time, and we must solve the above equations *implicitly* to calculate ω_n and ζ. [The equation is implicit since ω_n and ζ appear more than once, and inside exponential and sin functions.] Implicit equations are harder to solve than explicit equations.
  - Here is a brief summary of some of the methods we can use to solve for ω_n and ζ.
    - ***Exact solution***. We measure the damped (actual) natural frequency ω_d and then solve the above equations simultaneously for ω_n and ζ. Such a solution generally requires a simultaneous equation solver, such as EES, and good initial guesses. It also helps greatly to plot response vs. time.
    - ***Magnitude-only approximate solution***. This works best when ζ is *small*. The basis of this approximation is that the magnitude of the sine function is unity [sin(anything) varies between -1 and +1]. Thus, we ignore the sine term in the above equation, and consider only the decaying magnitude of the oscillations. Subtracting 1 from both sides and letting the magnitude of the sine function be unity, we get the approximate equation, $\boxed{\left|\dfrac{y - y_i}{y_f - y_i} - 1\right| \approx \left|-\mathrm{e}^{-\zeta\omega_n t}\dfrac{1}{\sqrt{1-\zeta^2}}\right|}$. This equation is still implicit in ω_n and ζ, but is much easier to solve than the exact equation. Furthermore, if we look only at *peaks* in the response, the equation becomes *exact* since the sine function is exactly 1 at any peak.
    - ***Log-decrement method***. This also works best when ζ is *small*, and is similar in principle to the magnitude-only approximation, but we restrict our observations to *peaks* in the response. A brief summary of the procedure is provided here.
      - First, we define the ***oscillatory component of the nondimensional output*** as $\boxed{y^* = 1 - \dfrac{y - y_i}{y_f - y_i}}$. This ensures that y^* oscillates about *zero*. [*Note*: If $y_f = 0$, we can simply use y instead of y^*.]
      - The amplitude of oscillation decays with time due to the damping, as sketched below. Here, y^*_i is the amplitude of peak i (i is an integer counting each peak), n is the number of cycles (peak-to-peak oscillation amplitude) being considered (n = 3 in the sketch), t is the time it takes for n cycles to occur, and T is the ***period of oscillation*** ($T = t/n$), namely, the time period of one cycle.



      - We define δ as the ***log decrement***, and, the following relations apply:

$$\boxed{\ln\left(\frac{y^*_i}{y^*_{i+n}}\right) = n\delta},\quad \boxed{\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}},\quad \boxed{\omega_d = \frac{2\pi}{T} = 2\pi f_d},\ \text{and}\ \boxed{\omega_n = 2\pi f_n = \frac{\omega_d}{\sqrt{1-\zeta^2}}}.$$

      - We use the first equation to calculate δ, then the second equation to calculate ζ. Note that damped natural physical frequency $f_d = 1/T$; ***this is the frequency observed in the experiment***.
      - The technique works regardless of what n we choose, and regardless of which peaks we choose.