Dimensional Analysis

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Primary Dimensions (Review)

- There are seven ***primary dimensions***. All other dimensions can be formed by combinations of these. The seven primary dimensions are, along with their symbols:

mass	length	time	temperature	current	amount of light	amount of matter
{m}	{L}	{t}	{T}	{I}	{C}	{N}

- The primary dimensions of variables in an experiment or analysis can be used to our advantage – *to reduce the required amount of effort*.

Dimensional Homogeneity

- We state the ***law of dimensional homogeneity*** as "Every additive term in an equation must have the same dimensions."
- Example: The total energy (E) of a system is composed of internal energy (U), kinetic energy (KE), and potential energy (PE), i.e., $E = U + \text{KE} + \text{PE}$.
- Let's look at the primary dimensions of each term in this equation:

$\{E\} = \{\text{energy}\} = \{\text{force} \cdot \text{length}\}$	$\{E\} = \{\text{mL}^2/\text{t}^2\}$
$\{U\} = \{mu\} = \left\{\text{mass}\dfrac{\text{energy}}{\text{mass}}\right\} = \{\text{energy}\}$	$\{U\} = \{\text{mL}^2/\text{t}^2\}$
$\{KE\} = \left\{\dfrac{1}{2}mV^2\right\} = \left\{\text{mass}\dfrac{\text{length}^2}{\text{time}^2}\right\}$	$\{KE\} = \{\text{mL}^2/\text{t}^2\}$
$\{PE\} = \{mgz\} = \left\{\text{mass}\dfrac{\text{length}}{\text{time}^2}\text{length}\right\}$	$\{PE\} = \{\text{mL}^2/\text{t}^2\}$

- The law of dimensional homogeneity is the basis for the useful technique of dimensional analysis, which we discuss next.

Dimensional Analysis and the Method of Repeating Variables

- Dimensional analysis is a simple, powerful tool that is useful in *all* disciplines (but unfortunately, is usually taught only in fluid mechanics).
- The ***goal*** of dimensional analysis is to reduce the number of independent variables in a problem.
- We accomplish this by converting all dimensional variables into ***nondimensional parameters***, called "Pi's", and given the symbol Π (upper case Greek letter pi).
- The ***Method of Repeating Variables*** has 6 steps:

Step 1: List the parameters in the problem and count their total number, n.

Step 2: List the primary dimensions of each of the n parameters.

Step 3: Set the *reduction*, j. As a first guess, set j as the number of primary dimensions. Calculate k, the expected number of Πs, $k = n - j$.

Step 4: Choose j *repeating parameters*.

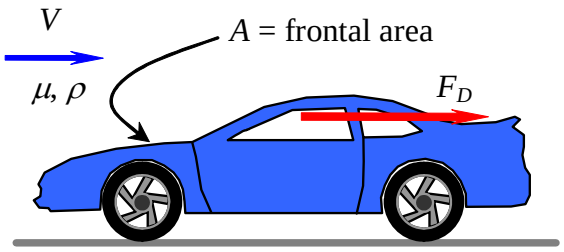
Step 5: Construct the k Πs (may need to reset j), and manipulate as necessary.

Step 6: Write the final functional relationship and check your algebra.

- The technique is best learned by example and practice. Let's look, for example, at measurement of the aerodynamic drag on a car.

- Example

Given: The drag force F_D on a car is a function of four variables: air velocity V, air density ρ, air viscosity μ, and the frontal area A of the car.



To do: Express this relationship in terms of nondimensional variables.

Solution: We follow the six steps above for the method of repeating variables.

- ***Step 1***: We list and count the variables.
  F_D = function of (V, ρ, μ, A). There are 5 variables, so $n = 5$.
- ***Step 2***: We list the primary dimensions of each variable:

  $\{F_D\} = \{mL/t^2\}$, $\{V\} = \{L/t\}$, $\{\rho\} = \{m/L^3\}$, $\{\mu\} = \{m/L\cdot t\}$, and $\{A\} = \{L^2\}$
- ***Step 3***: We set the reduction j. As our first guess, we set j equal to the number of primary dimensions represented in the problem. Here, we see mass (m), length (L), and time (t), so we set $j = 3$. [*Note*: If this does not work, we decrease j by 1 and start over from here.]
  We expect $k = n - j = 5 - 3 = 2$ Πs.
- ***Step 4***: We choose j ***repeating parameters***. Since $j = 3$, we choose V, ρ, and A. *Note*: This is the trickiest part of the process. How do we know which parameters to pick as the repeating parameters? Here are some guidelines or "rules" about picking the repeating parameters:

  1. Never pick the dependent variable (the one on the left) – in this case, we cannot pick F_D.
  2. The repeating variables cannot *by themselves* form a dimensionless group.
  3. All the primary dimensions in the problem must be represented by the j repeating parameters.
  4. Do not pick variables that are already dimensionless (e.g., angles).
  5. Do not pick two variables with the *same* dimensions or with dimensions that are *powers* of each other (e.g., cannot pick both a length {L} and an area {L^2} – in terms of dimensional analysis, these are really the same, since they both represent *only* a length).
  6. Pick "common" variables, since the repeating variables end up appearing in more than one Π. (That's why we call them "repeating variables" in the first place!)
  7. Whenever possible, choose *simple* variables (e.g., pick {L} instead of {mL^2t/T} if appropriate.)

- ***Step 5***: Generate the Πs:

  $\Pi_1 =$ dependent Pi $= F_D V^a \rho^b A^c$ [the dependent Pi contains the dependent variable on the left side]

$$\{\Pi_1\} = \{m^0 L^0 t^0\} = \text{dimensionless} = \left\{\frac{mL}{t^2}\left(\frac{L}{t}\right)^a \left(\frac{m}{L^3}\right)^b (L^2)^c\right\} = \{m^{1+b} t^{-2-a} L^{1+a-3b+2c}\}$$

Equate exponents: $m \rightarrow 0 = 1 + b \rightarrow b = -1$

$t \rightarrow 0 = -2 - a \rightarrow a = -2$

$L \rightarrow 0 = 1 + a - 3b + 2c \rightarrow c = -1$

Thus, $\Pi_1 = F_D V^a \rho^b A^c = F_D V^{-2} \rho^{-1} A^{-1} = \dfrac{F_D}{\rho V^2 A}$

We "manipulate" by multiplying by a constant (this is perfectly okay in dimensional analysis, since a constant has no dimensions, and we are not changing the nondimensional nature of the Π). We manipulate in order to get agreement with the commonly accepted and published dimensionless parameter called the ***drag coefficient***, i.e., we set $\boxed{\Pi_1 = C_D = \dfrac{F_D}{\frac{1}{2}\rho V^2 A}}$

Similarly, we generate the second Π by using the same three repeating variables, i.e., we set

$\Pi_2 =$ independent Pi $= \mu V^d \rho^e A^f$

$$\{\Pi_2\} = \{m^0 L^0 t^0\} = \text{dimensionless} = \left\{\frac{m}{L\cdot t}\left(\frac{L}{t}\right)^d \left(\frac{m}{L^3}\right)^e (L^2)^f\right\} = \{m^{1+e} t^{-1-d} L^{-1+d-3e+2f}\}$$

Equate exponents: m $\rightarrow$ $0 = 1 + e \rightarrow e = -1$

t $\rightarrow$ $0 = -1 - d \rightarrow d = -1$

L $\rightarrow$ $0 = -1 + d - 3e + 2f \rightarrow f = -1/2$

Thus, $\Pi_2 = \mu V^d \rho^e A^f = \mu V^{-1} \rho^{-1} A^{-½} = \dfrac{\mu}{\rho V \sqrt{A}}$

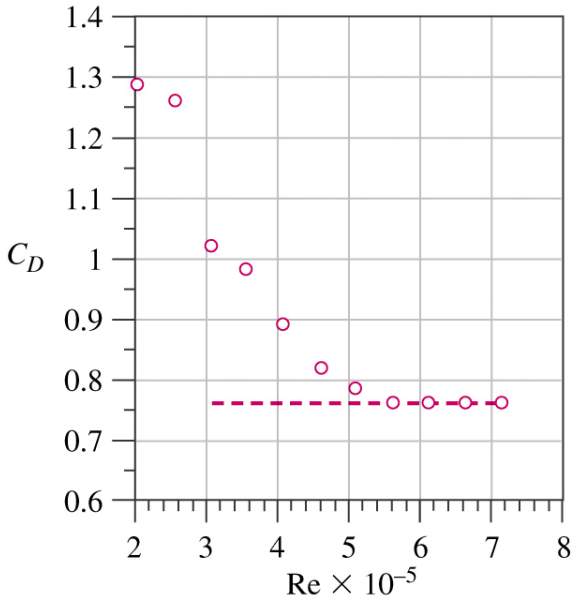
In this case, we "manipulate" by taking the inverse of the Π (we argue that since the Π is dimensionless, its inverse is also dimensionless). We do this so that our final nondimensional parameter is one of the standard, published dimensionless parameters, namely, the well-known ***Reynolds number***:

$\boxed{\Pi_2 = \text{Re} = \dfrac{\rho V \sqrt{A}}{\mu}}$ where we recognize that $\sqrt{A}$ is a length scale of the car.

- ***Step 6***: We write the final functional relationship between the Πs. In general, this relationship is of the form $\Pi_1 = \text{function}(\Pi_2, \Pi_3, ... \Pi_k)$ for *k* Πs. In this example, *k* = 2, so our final result is

$\boxed{\Pi_1 = \text{function}(\Pi_2)}$, i.e., $\boxed{\dfrac{F_D}{\frac{1}{2}\rho V^2 A} = \text{function}\left(\dfrac{\rho V \sqrt{A}}{\mu}\right)}$. Or, $\boxed{C_D = \text{function}(\text{Re})}$.

- What have we achieved? Well, let's compare the number of parameters in the original problem with that of the reduced (nondimensionalized) problem:
  - Original problem: F_D = function of (V, ρ, μ, A) $\rightarrow$ 5 parameters (drag force is a function of 4 variables).
  - Reduced problem: C_D = function of Re $\rightarrow$ 2 parameters (drag coefficient is a function of 1 nondimensional parameter, namely the Reynolds number).
- **We have reduced the number of independent parameters by 3, i.e., from 4 to 1!**
- This saves us significant time and money on experiments and/or computational analysis since we need to vary only one parameter (Reynolds number) rather than 4 parameters (velocity, density, viscosity, and frontal area). ***This is the power of dimensional analysis***.



- Finally, in some simple dimensional analysis problems, *k* = 1 (only *one* nondimensional parameter is formed). In Step 6, then, Π_1 = function(nothing). This makes sense only if Π_1 = constant. In such cases, the method of repeating variables produces an *equation*, Π_1 = constant, not merely a relationship between Πs. The equation is correct, but the constant is unknown, and must be found experimentally.