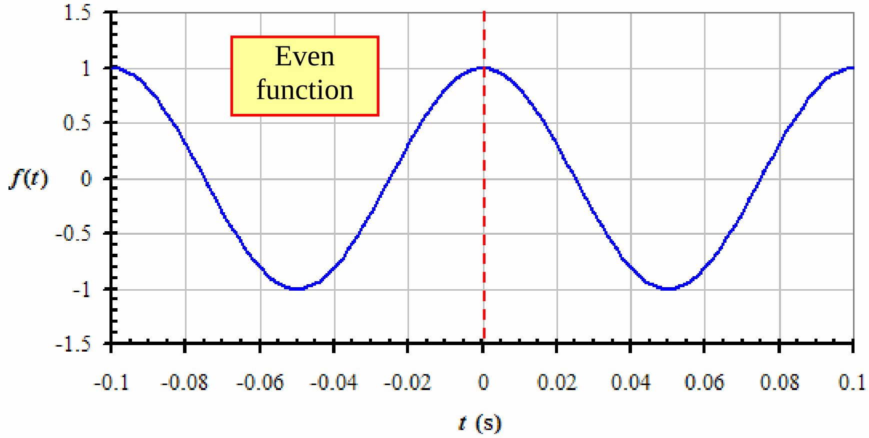
# Spectral Analysis (Fourier Series)

Author: John M. Cimbala, Penn State University
Latest revision: 19 February 2010

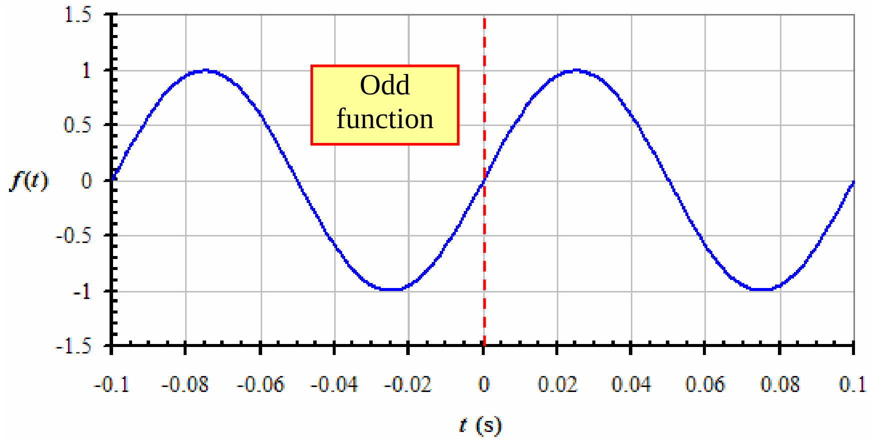
## Introduction

- There are many applications of ***spectral analysis***, in which we determine the *frequency content* of a signal.
- For analog signals, we use ***Fourier series***, which we discuss in this learning module.
- For digital signals, we use ***discrete Fourier transforms***, which we discuss in a later learning module.

## Even and odd functions

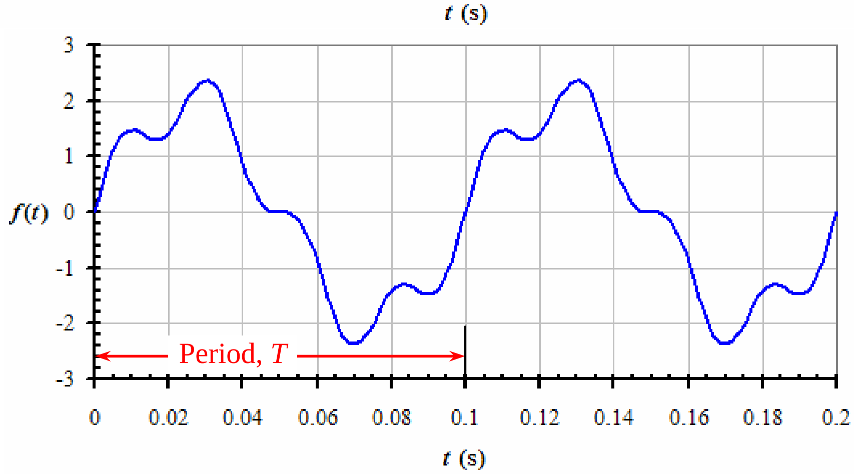



- An ***even function*** is one in which $f(-t) = f(t)$; *an even function is symmetric about t = 0*.
  - Consider a **cosine wave** as sketched to the right.
  - Notice that the function on the left is the mirror image of that on the right around $t = 0$.
  - Even functions are also called ***symmetric functions***.




- An ***odd function*** is one in which $f(-t) = -f(t)$; *an odd function is antisymmetric about t = 0*.
  - Consider a **sine wave** as sketched to the right.
  - Notice that the function on the left is the *negative* of the mirror image of that on the right around $t = 0$.
  - Odd functions are also called ***antisymmetric functions***.
- Most functions encountered in the laboratory are a combination of both even and odd functions.

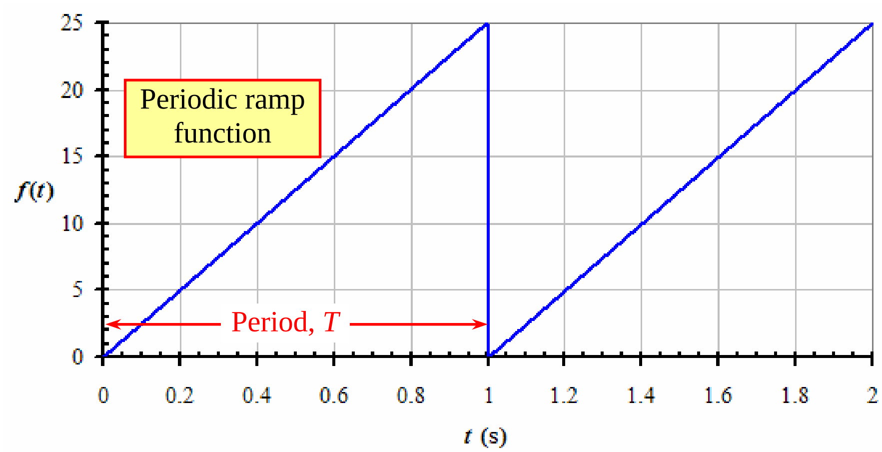
## Fourier series analysis




- Consider an arbitrary *periodic* function or signal, $f(t)$. Suppose the signal repeats itself over and over again at some ***period*** $T$, as sketched to the right.
- The ***fundamental frequency*** of the signal is defined as $\boxed{f_0 = 1/T}$; $f_0$ is also called the ***first harmonic frequency***. $f_0$ is the frequency at which the signal repeats itself.
- We also define the ***fundamental angular frequency***, $\boxed{\omega_0 = 2\pi f_0}$.
  $f_0$ has units of Hz (cycles/s), but $\omega_0$ has units of radians/s since there are $2\pi$ radians per cycle. Other names for fundamental angular frequency are ***fundamental radial frequency*** and ***fundamental radian frequency***.
- In a signal like that shown above, there may be more frequency components than just the fundamental one. In fact, there may be *many* other frequency components, called ***harmonics***, present in the signal.
- It turns out that *any periodic function f(t) can be expressed as the sum of a constant plus a series of sine and cosine terms representing the contribution of each harmonic*. Such a series is called a ***Fourier series***,
  $$\boxed{f(t) = c_0 + \sum_{n=1}^{\infty} a_n \sin(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \cos(n\omega_0 t)}.$$

- The first term on the right is a constant, which is simply the average of the function over the entire period $T$. The second collection of terms is the sine (odd) terms, and the third is the cosine (even) terms.
- The terms in the above equation are defined as $c_0 = \frac{1}{T}\int_0^T f(t)\,dt$ = average of $f(t)$ over time period $T$, $a_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)\,dt$ = coefficients of the sine terms, and $b_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)\,dt$ = coefficients of the cosine terms.
- If $f(t)$ is a voltage signal, you can think of $c_0$ as the ***DC offset*** of the signal. It turns out that $c_0$ is the same as $b_0/2$, where $b_0$ is evaluated at $n = 0$, making $\cos(n\omega_0 t)$ equal to 1 for all $t$ over the integral.
- For an *even* function, all the $a_n$ coefficients are zero. Since only the cosine terms remain, we call this a ***Fourier cosine series***.
- For an *odd* function, a ***Fourier sine series,*** all the $b_n$ coefficients are zero (only the sine terms remain).

- **Example**:
***Given:*** A periodic ramp function, $f(t) = Gt$ from $t = 0$ to 1 s, where $G = 25$ V/s. The function has units of volts, and is periodic with period $T = 1$ s.




***To do:*** (***a***) Calculate $f_0$ and $\omega_0$, (***b***) calculate $c_0$, $a_1$, $a_2$, $b_1$, and $b_2$, the first few Fourier series coefficients, and (***c***) Discuss how well the Fourier series approximation agrees with the original signal.

***Solution:***

(***a***) Calculate $f_0$ and $\omega_0$.

- We plot the function to the right, repeating the pattern every 1 s since the fundamental period is $T = 1$ s.
- The fundamental frequency is $f_0 = 1/T = 1/(1\text{ s}) = 1$ Hz; $\boxed{f_0 = 1\text{ Hz}}$.
- The fundamental angular frequency is then $\omega_0 = 2\pi f_0 = 2\pi\left(1\frac{1}{\text{s}}\right) = 2\pi\frac{\text{radians}}{\text{s}}$; $\boxed{\omega_0 = 2\pi\frac{\text{rad}}{\text{s}}}$.

(***b***) Calculate $c_0$, $a_1$, $a_2$, $b_1$, and $b_2$, the first few Fourier series coefficients.

- We calculate $c_0 = \frac{b_0}{2} = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{T}\int_0^T 25\frac{\text{V}}{\text{s}}t\,dt = \left[\frac{1}{T}\frac{25}{2}\frac{\text{V}}{\text{s}}t^2\right]_{t=0}^{t=T}$. For $T = 1$ s, $\boxed{c_0 = 12.5\text{ V}}$.

  Note that this could have been done by inspection, since the average value of the ramp function between 0 and $T$ is simply halfway between its two extremes, 0 and 25 V, or 12.5 V.
- We calculate $a_1 = \frac{2}{T}\int_0^T f(t)\sin(\omega_0 t)\,dt = \frac{2}{T}\int_0^T 25\frac{\text{V}}{\text{s}}t\sin\left(2\pi\frac{\text{rad}}{\text{s}}t\right)dt$, which can be solved by integration by parts, or by looking it up in a math book, CRC manual, webpage, etc.
- The following formula is useful for this integration: $\boxed{\int x\sin(px)dx = \frac{1}{p^2}\sin(px) - \frac{x}{p}\cos(px)}$. Setting constant $p = 2\pi$ rad/s yields $a_1 = \frac{2}{1\text{ s}}\left(25\frac{\text{V}}{\text{s}}\right)\left[\frac{1}{4\pi^2\text{ rad}^2/\text{s}^2}\sin\left(2\pi\frac{\text{rad}}{\text{s}}t\right) - \frac{t}{2\pi\text{ rad/s}}\cos\left(2\pi\frac{\text{rad}}{\text{s}}t\right)\right]_{t=0}^{t=1\text{ s}}$ =

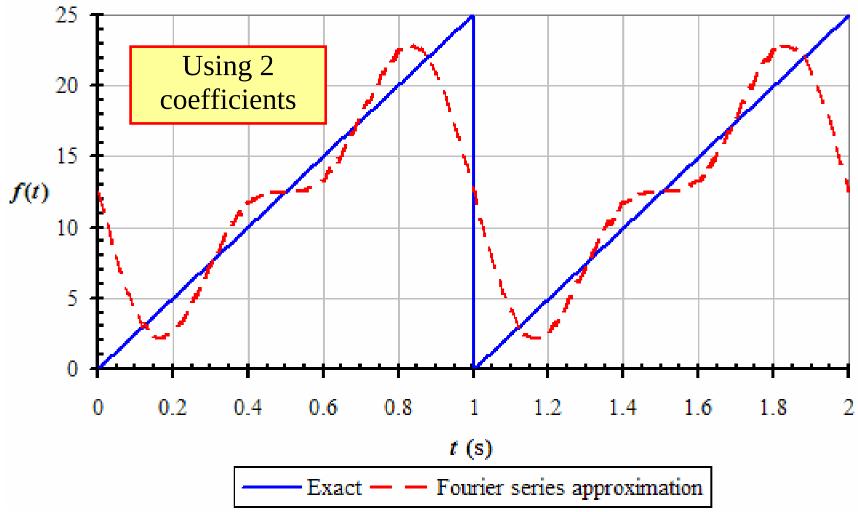
  $(50\text{ V})\left[\frac{1}{4\pi^2}(\sin(2\pi)-\sin(0)) - \frac{1}{2\pi}(\cos(2\pi)-0)\right]$ = $(50\text{ V})\left[\frac{1}{4\pi^2}(0-0) - \frac{1}{2\pi}(1-0)\right]$ = $-\frac{25}{\pi}$ V;

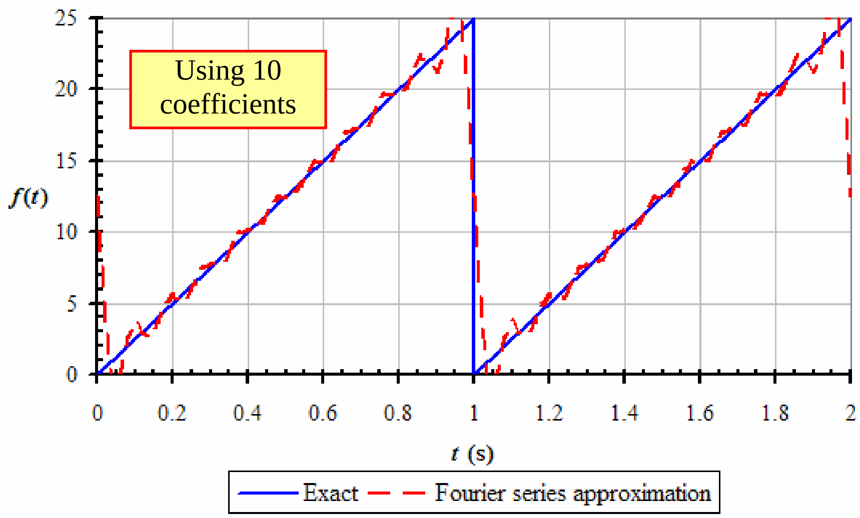
  $\boxed{a_1 = -\frac{25}{\pi}\text{ V} = -7.95775\text{ V}}$ (to 6 significant digits).

- o Similarly, we integrate to find $a_2$, using the same integral formula as above, except that for $n$ = 2, the value inside the cosine and sine terms is $4\pi t$ instead of $2\pi t$. The result is $\boxed{a_2 = -\frac{25}{2\pi}\text{ V} = -3.97887\text{ V}}$.
- o It turns out that all the $b_n$ coefficients are zero! You can prove this by integrating in the same fashion as above, using the integral formula $\boxed{\int x\cos(px)dx = \frac{1}{p^2}\cos(px) + \frac{x}{p}\sin(px)}$.
- o Alternately, we recognize by inspection of the original function, that if the entire function is shifted downward by the value of $c_0$, the function becomes an odd function. This implies that, except for the constant $c_0$, the even terms of the Fourier series are all zero. Thus all the $b_n$ coefficients are zero.
- o The final form of the Fourier series is thus simply a ***Fourier sine series***, $f(t) = c_0 + \sum_{n=1}^{\infty} a_n \sin(n\omega_0 t)$, or $\boxed{f(t) = 12.5\text{ V} - (7.95775\text{ V})\sin\left(2\pi\frac{\text{rad}}{\text{s}}t\right) - (3.97887\text{ V})\sin\left(4\pi\frac{\text{rad}}{\text{s}}t\right) + \ldots}$, where only the first two $a_n$ coefficients have been included here.

(***c***) Discuss how well this Fourier series approximation agrees with the original signal.

- o A plot is shown to the right.
- o It is clear that the Fourier series approximation (red dashed curve) with only a constant plus two sine function coefficients does not do such a great job at duplicating the original ramp function (solid blue curve).
- o This indicates that the original signal contains significant contributions from higher harmonics as well. Note that the first harmonic is the fundamental frequency (1 Hz), and the second harmonic is twice that, or 2 Hz. These are the only two harmonics taken into account in the above representation.
- o If we include *more coefficients*, the Fourier series approximation does a better job. For example, shown to the right is the plot for the case with 10 coefficients ($a_1$ through $a_{10}$).
- o There are still some wiggles, especially near the sharp corners at $t$ = 0, 1, 2 s, etc., but the overall agreement is much better.



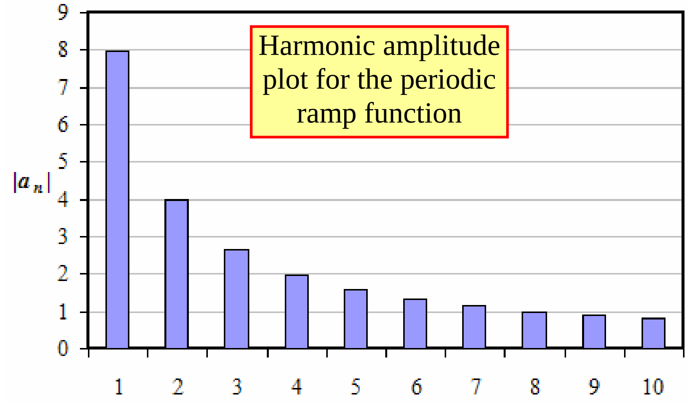

***Discussion:*** In an example like this one where there is a sudden jump, we must calculate the coefficients of a large number of harmonics in order to get good duplication of the original signal.

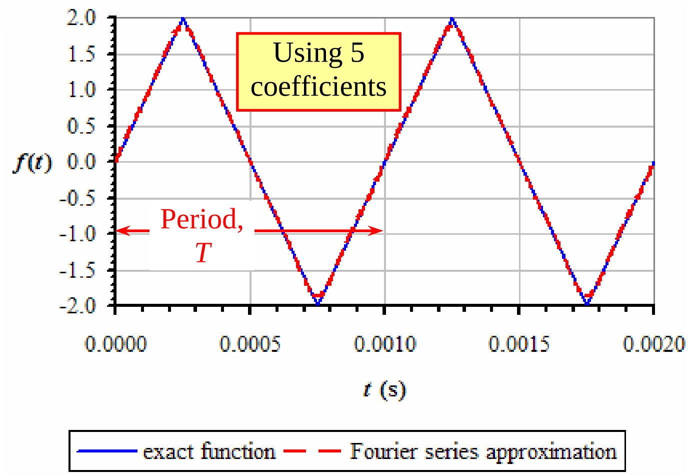
**Numerical calculation of the Fourier coefficients**

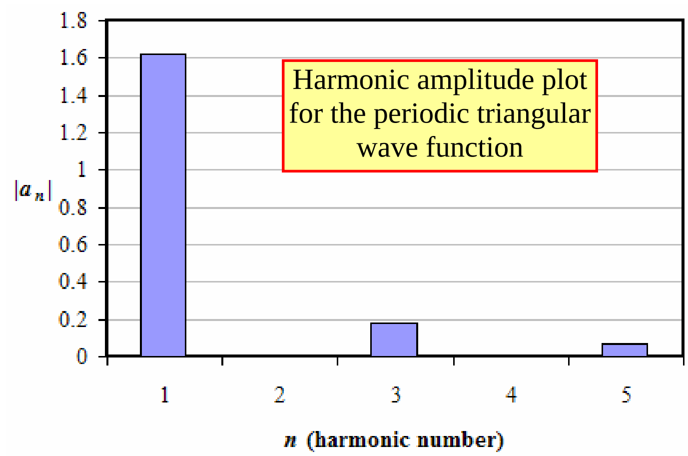
- The integration required to find the Fourier coefficients is often quite tedious - in some cases impossible analytically.
- Several numerical integration schemes have been developed to integrate these equations, but these are beyond the scope of this course.

## Harmonic amplitude plots

- The *relative importance* of each harmonic in a Fourier series is compared by plotting the RSS amplitude of each sine and cosine coefficient $\sqrt{a_1^2+b_1^2}$, $\sqrt{a_2^2+b_2^2}$, $\sqrt{a_3^2+b_3^2}$, ... versus harmonic number $n$ = 1, 2, 3, ... Such a plot is called a ***harmonic amplitude plot***; ***it shows the frequency content of the signal***.
- In the simplified case of a Fourier *sine* series, the RSS amplitudes reduce to $|a_1|$, $|a_2|$, $|a_3|$, … since all the cosine coefficients $b_n$ are zero.
- For the ramp function example above, the harmonic amplitude plot is shown to the right.
- From this plot, we see that the relative importance of the harmonics decreases steadily. However, in this particular example, the decay of amplitude is not very fast.
- For some other periodic functions (especially *smoother* ones), the decay with $n$ is much faster, and the signal can be duplicated quite well with only a few of the coefficients of the Fourier series included.
- Consider, for example, a triangular waveform. This time the fundamental frequency is 1000 Hz, and the peak to peak amplitude is –2.0 to 2.0.
  - Here again all the cosine terms are zero (this is also a ***Fourier sine series***).
  - The first five sine coefficients are calculated. It turns out that only the odd $a_n$ coefficients are non-zero; the even ones like $a_2$, $a_4$, etc. are all zero.
  - A plot comparing the original function to the Fourier sine series representation is shown to the right.
  - It is hard to distinguish between the two curves, except at the sharp corners, indicating that the amplitudes of the harmonics drop off rapidly with $n$. This can be seen in the harmonic amplitude plot to the right.
  - Note the *rapid* decay of amplitude with $n$. The first five coefficients do an excellent job of representing this function (but the sharp corners are still rounded off, as with all Fourier series approximations).
  - Only by calculating *all* the Fourier coefficients ($n \to \infty$) can we duplicate the original signal *exactly*, including the sharp corners.




## Fourier transforms

- For an analog (continuous signal), we can mathematically extend the summations of the Fourier series into *integrations*. The resulting transformations are called ***Fourier transforms***.
- Fourier transforms are more useful than Fourier series, and are discussed in the next learning module.