Today, we will:

- Do some review example problems – digital data acquisition, aliasing, FFTs
- Discuss the concept of **leakage** in more detail.

Example: DFTs and FFTs

Given: Vibrations are observed in a computer hard drive. Preliminary measurements with an accelerometer indicate that the strongest vibrations occur at a fundamental frequency of about 7200 rpm (120 Hz), and the engineers expect that significant harmonics may be present up to the 4th harmonic. Data are acquired with a digital data acquisition system.

To do:

(***a***) What is the minimum sampling frequency they should use to avoid aliasing?

(***b***) They decide to sample data at f_s = 4000 Hz. They take 1024 data points (N = 1024 pts) and plug the data into a computer program to calculate a DFT (or FFT). How many *useful* data points (frequencies) will appear on their frequency spectrum?

(***c***) The results are good, but the frequency resolution is not as good as they had hoped. Bob and Ted argue about what they should do to improve the frequency resolution:

- Bob suggests using the same N, but increasing the sampling frequency f_s.
- Ted suggests using the same f_s, but increasing the number of data points N.

Which of these suggestions would improve the frequency resolution and why?

Solution:

(a) • $f_0 = 120$ Hz (fundamental frequency, or first harmonic)

(To meet the Nyquist criterion)

• 4th harmonic = $f = 4f_0 = 480$ Hz

• So, to avoid aliasing, must sample at $f_s > 2(480\text{ Hz})$, or $\boxed{f_s > 960\text{ Hz}}$

(b) • Set $f_s = 4000$ Hz, $N = 1024$ data pts.

• But, in plotting the frequency spectrum, we throw out all frequencies larger than the folding frequency — i.e. we throw out half of the FFT values. [Note: we use all 1024 data pts to do the FFT, but we use only the first half of the FFT output]

• So, 1024/2 = $\boxed{\text{512 useful data pts on the frequency spectrum.}}$

OR, since we include both the DC component ($f=0$) & $f_{folding}$, $\boxed{\text{513 is the actual answer}}$

(c) $\boxed{\text{Ted is right}}$ $\boxed{\Delta f = \frac{1}{T}} = \frac{f_s}{N}$ ← if $f_s \uparrow$, $\Delta f \uparrow$; ← if $N \uparrow$, $\Delta f \downarrow$

[We could also decrease f_s to, say, 1000 Hz]

Since we want Δf to be smaller, Bob's idea would make it worse!

Example: FFTs

Given: A voltage signal has a 1.50 V DC component and two periodic components:

- Frequency f_1 = 115 Hz, amplitude A_1 = 2.00 V
- Frequency f_2 = 540 Hz, amplitude A_2 = 0.500 V

There is also some noise. We sample the signal digitally at 1000 Hz, taking 256 data points.

(*a*) To do: Is there any aliasing? If so, calculate f_a.

(*b*) To do: Calculate the frequency resolution and sketch the frequency spectrum.

Solution:

(a). $f_s = 1000$ Hz. Nyquist says that we can reliably measure frequency components up to half of f_s → so, $\boxed{f \text{ must be} < 500 \text{ Hz to avoid aliasing}}$

- Here, f_1 is okay ($f_1 = 115$ Hz < 500 Hz) → $\boxed{f_1 \text{ will not alias}}$
  But f_2 is not okay ($f_2 = 540$ Hz $\cancel{<}$ 500 Hz) → $\boxed{f_2 \text{ will alias}}$
- Use our eq. to calculate f_a.

$$f_a = \left| f - f_s * \mathrm{NINT}(f/f_s) \right|$$
$$= \left| 540 - 1000 * \mathrm{NINT}(\underbrace{540/1000}_{(0.540)}) \right|$$
$$= \left| 540 - 1000 * 1 \right| = 460$$

$$\boxed{f_a = 460 \text{ Hz}}$$

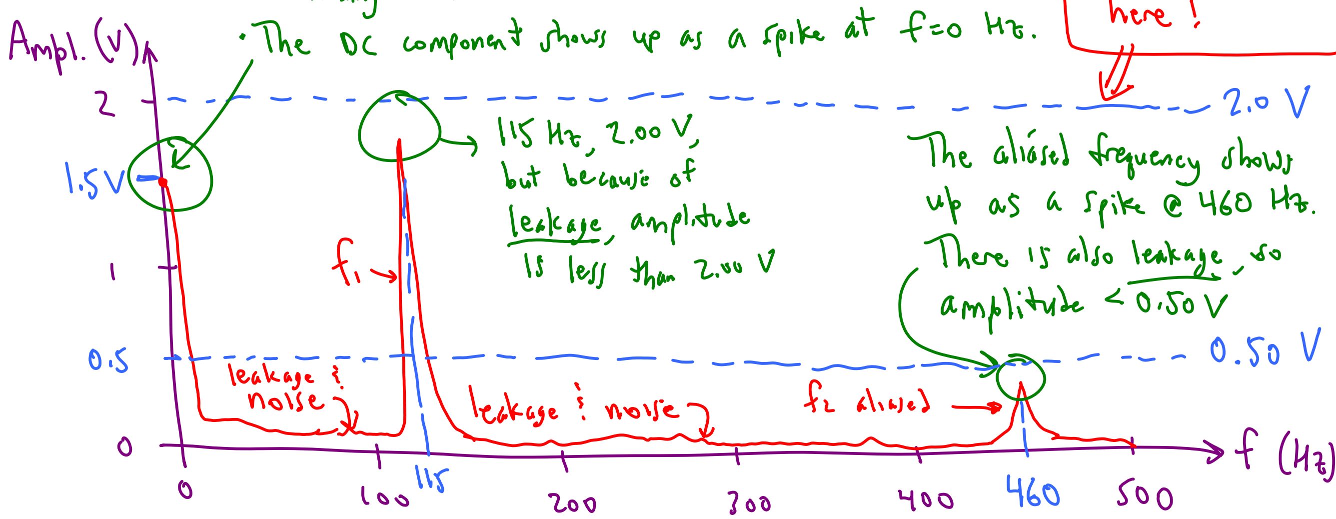
(b) $\Delta f = \dfrac{1}{T} = \dfrac{f_s}{N} = \dfrac{1000 \text{ pts/s}}{256 \text{ pts}} = 3.90625 \rightarrow \boxed{\Delta f = 3.90625 \text{ Hz}}$

(not very good frequency resolution)

Sketch the frequency spectrum:

- We plot up to $f_{folding} = f_s/2 = 1000/2 = 500$ Hz; ignore the rest. ☆

There is both leakage and aliasing here!



Example: FFTs

Given: Brian samples a voltage signal digitally at a sampling frequency of 400 Hz. He plugs the data into Excel, calculates an FFT, and plots the frequency spectrum.

(*a*) To do: If Brian samples data for 5.12 seconds, how many *useful* frequencies are plotted on the frequency spectrum?

(*b*) To do: If Brian samples data for 5.00 seconds, how many *useful* frequencies are plotted on the frequency spectrum?

(*c*) To do: For the situation of Part (*b*), what is the frequency resolution of the resulting frequency spectrum produced by Excel? *Give your answer to three significant digits.*

Solution:

(a) $T = 5.12\text{ s}$, $f_s = 400\text{ Hz} \rightarrow N = f_s T = \left(400\,\frac{1}{s}\right)(5.12\text{ s}) = \underline{2048 \text{ pts} = N}$

But, although all 2048 pts are used to perform the FFT, we throw away half of the FFT output to generate the frequency spectrum.

(we throw away the half for which $f > f_{folding} = 400/2 = 200\text{ Hz}$.)

Answer: 1024 useful frequencies (2048/2)

$\boxed{\text{Actually 1025 " " since we count both DC } (f=0) \text{ \& } f_{folding} (f=200)}$

(b) $T = 5.00\text{ s}$, $f_s = 400\text{ Hz} \rightarrow N = f_s T = \left(400\,\frac{1}{s}\right)(5.00\text{ s}) = \underline{2000 \text{ pts} = N}$

- BUT → Excel's FFT macro is limited to powers of 2 (4, 8, 16, 32 ... 1024, 2048..)
- So, we cannot use all 2000 data points. Instead we truncate to $\underline{N = 1024}$

Then, half (plus 1) of these are used for the frequency spectrum plot.

Answer $\boxed{\text{513 useful frequencies in the spectrum plot}}$

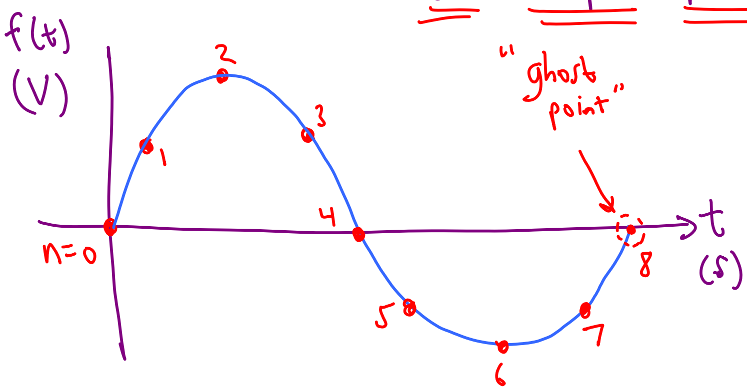
(c) $\Delta f = \frac{1}{T} = \frac{1}{5.00} = \underline{0.20\text{ Hz}}$ ✗ No → we truncated to 1024 pts!

- So, we actually use $T = \frac{N}{f_s} = \frac{1024\text{ pts}}{400\text{ pts/s}} = \underline{2.56\text{ s} = T}$ (useful data)
- Thus, actual $\Delta f = \frac{1}{T} = \frac{1}{2.56\text{ s}} = \boxed{0.3906\text{ Hz} = \Delta f}$

This is our actual frequency resolution

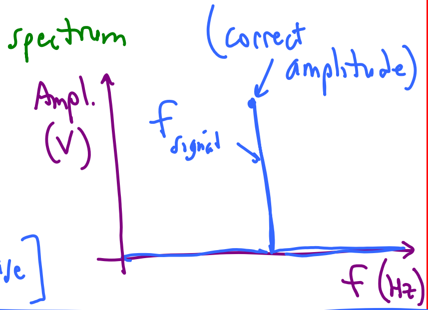
More notes about leakage:

- **Ideal situation** → Pure sine wave, sample 8 pts ($N=8$) & sample <u>one complete period</u>.



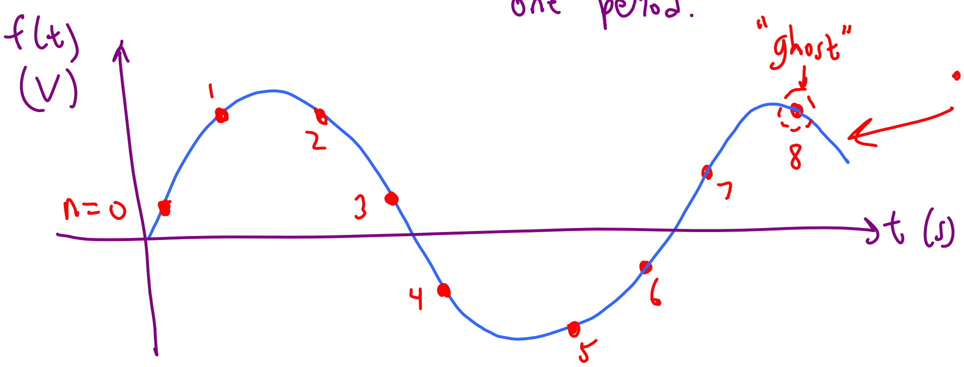
- We use data points 0 to 7
- We do not use point 8
- Point 8 is assumed to be identical to point 0.
- In this "perfect" case, point 8 is indeed the same as point 0.

- An FFT of these data yields a <u>perfect</u> frequency spectrum
  - No leakage
  - correct amplitude
  - max. amplitude (spike) occurs at exactly f_{signal}



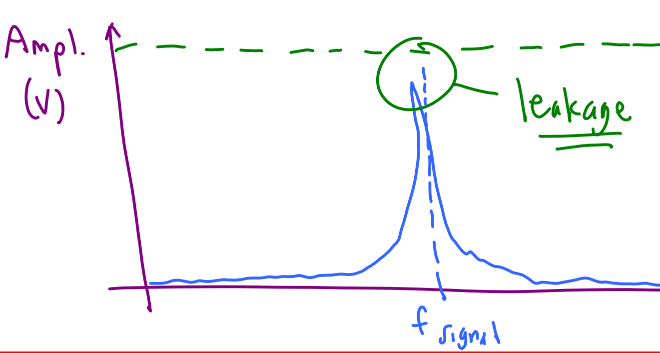
[First & last (ghost) data point are at the same voltage & phase]

- **More realistic situation** — we sample at a higher frequency, not exactly one period.



- Now, pt. 8 & point 0 are <u>not</u> at the same phase & <u>not</u> at the same voltage.

- But, the FFT assumes that point 8 is identical to point 0.
- This leads to <u>leakage</u>



Actual amplitude

f (Hz)

<u>leakage</u> →
- wrong amplitude
- wider spike (not a perfect spike)
- max amplitude at incorrect frequency