Today, we will:

- Review the pdf module: **Dynamic System Response (1st-order systems)**
- Do some example problems – dynamic system response for first-order systems

Example: First-order dynamic system response

Given: A first-order low-pass filter with R = 100 kΩ and C = 0.010 μF

(*a*) To do: Calculate the time constant τ and the static sensitivity K of this system.
(*b*) To do: Discuss how time constant τ is related to the cutoff frequency of the filter.
(*c*) To do: For a sudden change in input voltage, how long will it take for the nondimensional output to reach 99% of its final value?
(*d*) To do: If y_i = 1 V and y_f = 3 V, calculate y when t = 10.0 ms.

Solution:

(a) • From the learning module, we write the 1st-order ODE for a low-pass filter:

$$C\frac{dV_{out}}{dt} + \frac{1}{R}V_{out} = \frac{1}{R}V_{in}$$

• Compare to the standard form for a 1st-order dynamic system

$$a_1\frac{dy}{dt} + a_0 y = bx$$

(also from the learning module)

$a_1 = C$ (capacitance)

$y = V_{out}$ (output voltage = response)

$a_0 = \frac{1}{R}$

$b = \frac{1}{R}$

$x = V_{in}$ (input)

• By definition, $K = \frac{b}{a_0} = \frac{1/R}{1/R} = 1$ $\boxed{K = 1 = \text{static sensitivity}}$

[as $t \to \infty$, $V_{out} \to V_{in}$ when V_{in} = constant (DC voltage input)]

$\tau = \frac{a_1}{a_0} = \frac{C}{1/R} = RC$ $\boxed{\tau = RC = \text{1st-order time constant}}$

• Numbers: $\tau = (100{,}000\ \Omega)(0.01 \times 10^{-6}\ \text{F})\left(\frac{\text{C}}{\text{V}\cdot\text{F}}\right)\left(\frac{\text{A}\cdot\text{s}}{\text{C}}\right)\left(\frac{\text{V}}{\text{A}\cdot\Omega}\right) = 0.0010\ \text{s}$

$\boxed{\tau = 1.0\ \text{ms}}$

(b) Compare τ to cutoff frequency of the filter:

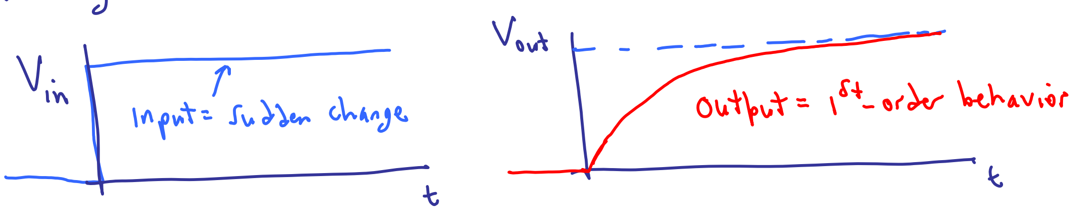
- Recall, $f_{cutoff} = \dfrac{1}{2\pi RC} = \dfrac{1}{2\pi\tau}$ $\qquad \boxed{f_{cutoff} = \dfrac{1}{2\pi\tau}}$

  $\omega_{cutoff} = \dfrac{1}{RC} = \dfrac{1}{\tau}$ $\qquad \boxed{\omega_{cutoff} = \dfrac{1}{\tau}}$

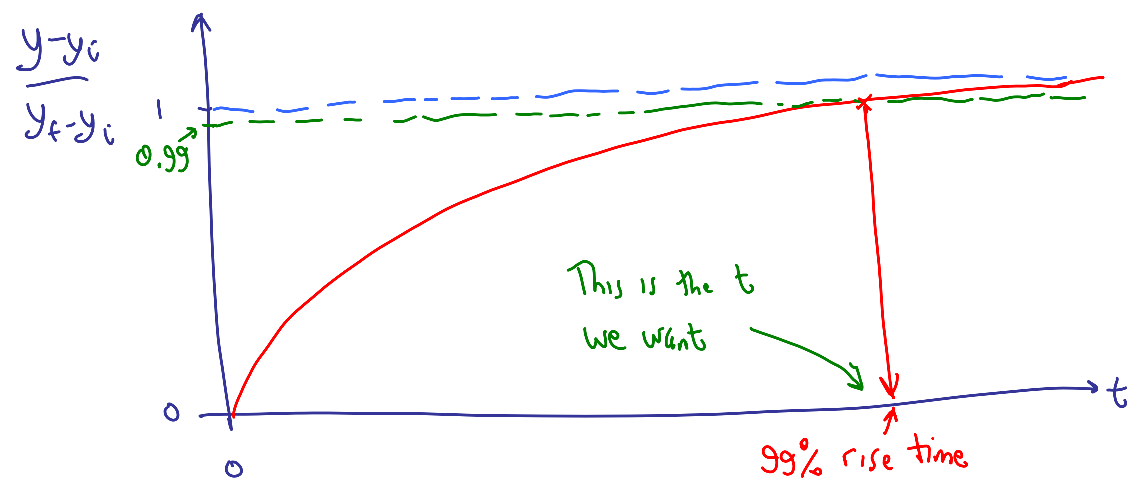
  $RC \rightarrow \tau$

- numbers: $\boxed{f_{cutoff} = \dfrac{1}{2\pi(0.0010\text{ s})} = 159.\text{ Hz}}$

(c) How long to reach 99% of final value?



- Use the nondimensional form of the solution (applies to any sudden change)



- Equation from the learning module, nondimensional [holds for any y_i & y_f]

$$\boxed{\frac{y-y_i}{y_f-y_i} = 1 - e^{-t/\tau}}$$

Set to 0.99

$\rightarrow$ Solve for t: $0.99 = 1 - e^{-t/\tau}$

$0.01 = e^{-t/\tau}$

$\ln(0.01) = -t/\tau$ $\qquad [\ln e^{\xi} = \xi]$

$\boxed{t = -\tau \ln(0.01)} = -(0.0010\text{ s})\ln(0.01)$

$\boxed{t = 0.00461\text{ s}}$

NOTE: Since $\tau = 0.0010$ s, this t is 4.61 time constants, close to 5 time constants

· Most people round to 5 @ 99% → A first-order system needs about 5 time constants to get within 99% of the final change of response

[For a sudden change]

(d) If $y_i = 1\,V$ & $y_f = 3\,V$, calculate y @ $t = 10$ ms

· Again, use the general nondimensional equation since it applies to any y_i & y_f

$$\frac{y - y_i}{y_f - y_i} = 1 - e^{-t/\tau} \rightarrow y = y_i + (y_f - y_i)\left(1 - e^{-t/\tau}\right)$$

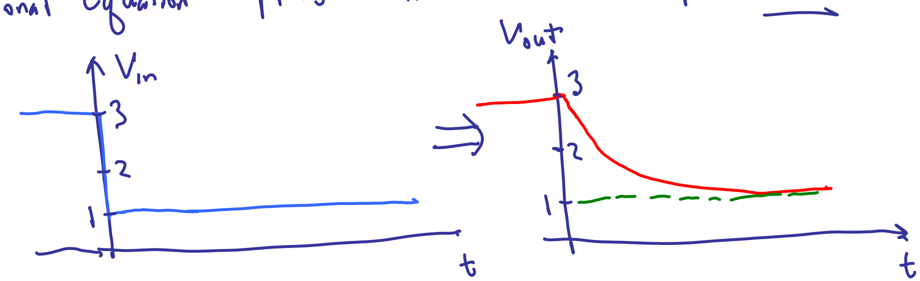
· Numbers: $$y = 1\,V + (3V - 1V)\cdot\left(1 - e^{-\frac{10\text{ ms}}{1\text{ ms}}}\right) = 2.999909\ V$$

$$\boxed{\simeq 3.00\ V}$$

[After 10 time constants, we have "reached" the final response output to the 5th digit]

· The general nondimensional equation applies even when the step is down

· E.g. Let $y_i = 3\,V$
$y_f = 1\,V$



· At $t = 1.0$ ms, what is y?

$$\frac{y - y_i}{y_f - y_i} = 1 - e^{-t/\tau} \rightarrow y = y_i + (y_f - y_i)\left(1 - e^{-t/\tau}\right)$$

$$y = 3V + (1V - 3V)\cdot\left(1 - e^{-\frac{1\text{ ms}}{1\text{ ms}}}\right)$$

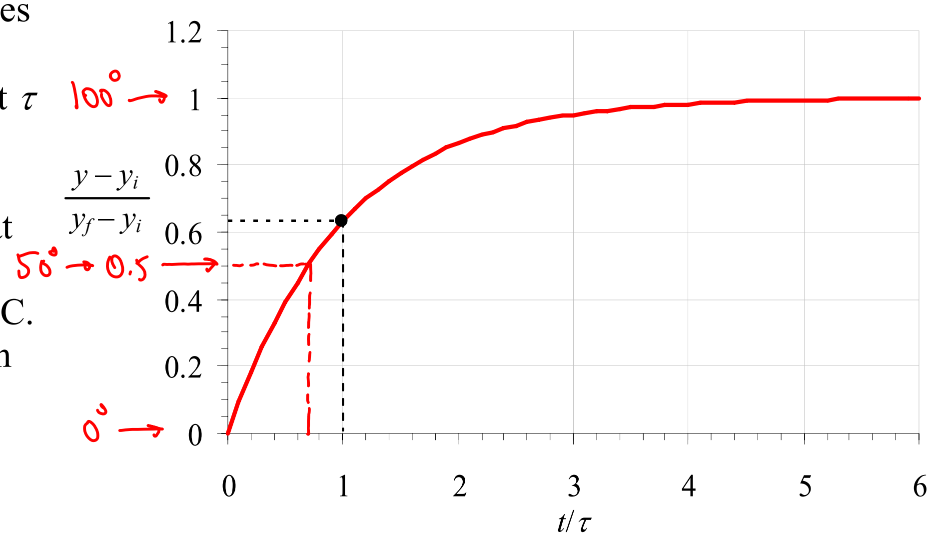
$$y = 1.7358\quad V$$

[After t = one time constant ($t = \tau$), y drops from 3 V to 1.7358 V]

Example: First-order dynamic system response

Given: A thermometer behaves as a first-order dynamic system with time constant τ = 1.00 s. At t = 0, the thermometer is plunged from a tank of ice water at T = 0°C into a tank of boiling water at T = 100°C.

To do: Calculate how long (in seconds) it takes for the thermometer to read 50°C.



Solution:

$$\frac{y-y_i}{y_f-y_i}=1-e^{\frac{-t}{\tau}}$$

Set this to 0.5 & solve for t

@ T = 50°, $\frac{y-y_i}{y_f-y_i} = \frac{50-0}{100-0} = \underline{0.5}$

$$0.5 = 1 - e^{-t/\tau}$$

$$-0.5 = -e^{-t/\tau}$$

$$\ln(0.5) = \ln\left(e^{-t/\tau}\right) = -t/\tau$$

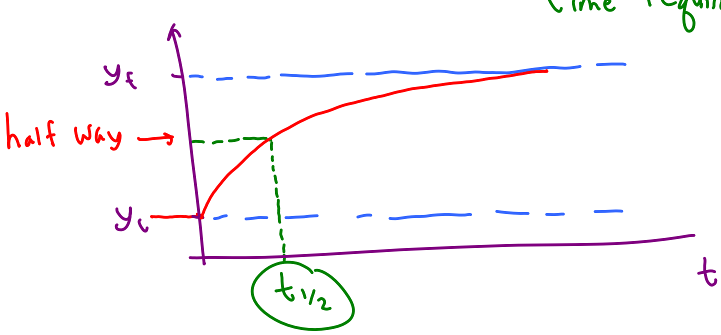
τ = time constant = given

$$t = -\tau \ln(0.5) = -(1.00\text{ s}) \ln(0.5) = 0.69315 \text{ s}$$

$$\boxed{t \approx 0.693 \text{ s}}$$

By the way, this is called the $\boxed{\text{half-life } t_{1/2}}$

time required to go half way from y_i to y_f



Example – A practical example of 1st-order dynamic system analysis

Given: A fire has occurred in a hotel room with volume $V = 85.\ \text{m}^3$. Immediately after the fire is extinguished, the mass concentration of hydrogen cyanide (HCN) is $c_i = 10{,}000\ \text{mg/m}^3$. The firemen blow "fresh" air into the room at $\dot{V} = 28.3\ \text{m}^3/\text{min}$. Since there is some smoke outside the building, the "fresh" air actually has an ambient mass concentration of HCN equal $c_a = 1.0\ \text{mg/m}^3$. The air is considered safe when the mass concentration of HCN in the room drops below $c = 5\ \text{mg/m}^3$.

To do: Calculate how long the firemen need to wait before entering the room.

Solution: Room ventilation is modeled by using a first-order ODE for mass concentration c in the room:

$$\boxed{\frac{dc}{dt} = -\frac{\dot{V}}{V}c + \frac{\dot{V}}{V}c_a}$$

• Compare to standard form:

$$a_1\frac{dy}{dt} + a_0 y = b x$$

$$1\frac{dc}{dt} + \frac{\dot{V}}{V}c = \frac{\dot{V}}{V}c_a$$

$a_1 = 1$

$a_0 = \dot{V}/V$

$b = \dot{V}/V$

$y = c$

$x = c_a$

• $\tau = \dfrac{a_1}{a_0} = \boxed{\dfrac{V}{\dot{V}} = \tau}$ ∴ nondimensional equation is $\boxed{\dfrac{y - y_i}{y_f - y_i} = 1 - e^{-t/\tau}}$

• Solve for t → $t = -\dfrac{V}{\dot{V}}\ln\left(1 - \dfrac{y - y_i}{y_f - y_i}\right) = -\dfrac{V}{\dot{V}}\ln\left(1 - \dfrac{c - c_i}{c_f - c_i}\right)$

where c = the "safe" concentration that we desire (5)

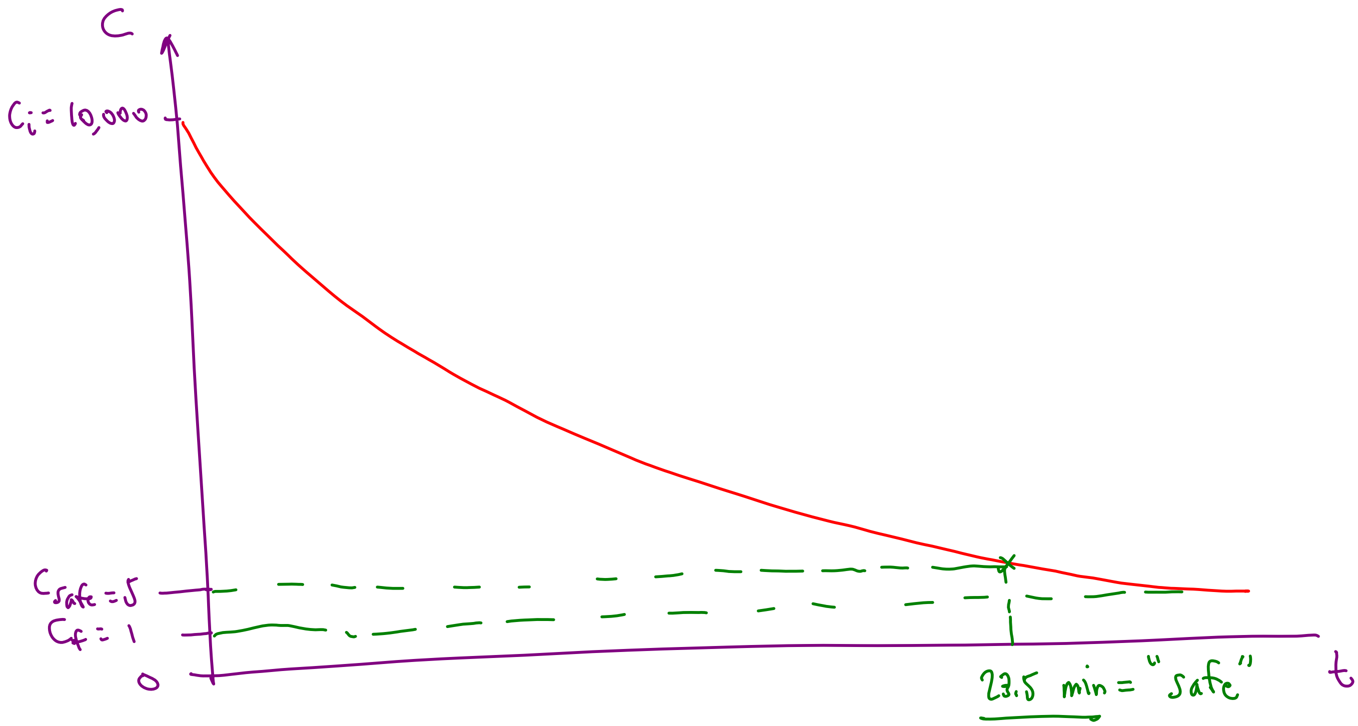
$c_f = c_a$ = ambient concentration [as $t \to \infty$, $c \to c_a$ = ambient concentration] (1)

c_i = initial concentration in the room (10,000)

• Numbers: $t = \dfrac{-85.\ \text{m}^3}{28.3\ \text{m}^3/\text{min}}\ln\left(1 - \dfrac{5 - 10{,}000}{1 - 10{,}000}\right) = 23.5$ minutes

Answer to 2 digits → $\boxed{t \approx 24.\ \text{minutes}}$

[The firemen must wait ≈ ½ hr. to enter safely]



Note: $\tau = \dfrac{\forall}{\dot{\forall}} = \dfrac{85\ m^3}{28.3\ m^3/min} = 3.0035\ min$

Need $\approx \dfrac{23.5}{3.0035} = 7.8\ \tau$'s

The room needs $\approx$ 8 time constants for c to decrease to safe levels