Today, we will:

- Discuss **half-life** in first-order dynamic systems
- Talk about first-order dynamic systems with a **ramp function input**
- Finish reviewing the pdf module: **Dynamic System Response (2nd-order systems)**
- Do some more example problems – dynamic system response

Definition of half-life: ***Half-life*** is the time required for a variable to go half-way from its present value to its final value. Half-life can also be thought of as the 50% response time.

- Most common & popular use is with <u>radioactive decay</u>

  e.g. <u>Carbon-14</u> decays to Nitrogen-14

  radioactive isotope of carbon (normally Carbon-12). C-14 is produced by cosmic radiation interacting with the upper atmosphere

  A small fraction of the C in CO_2 and in the bodies of all living creatures is C-14 instead of C-12.

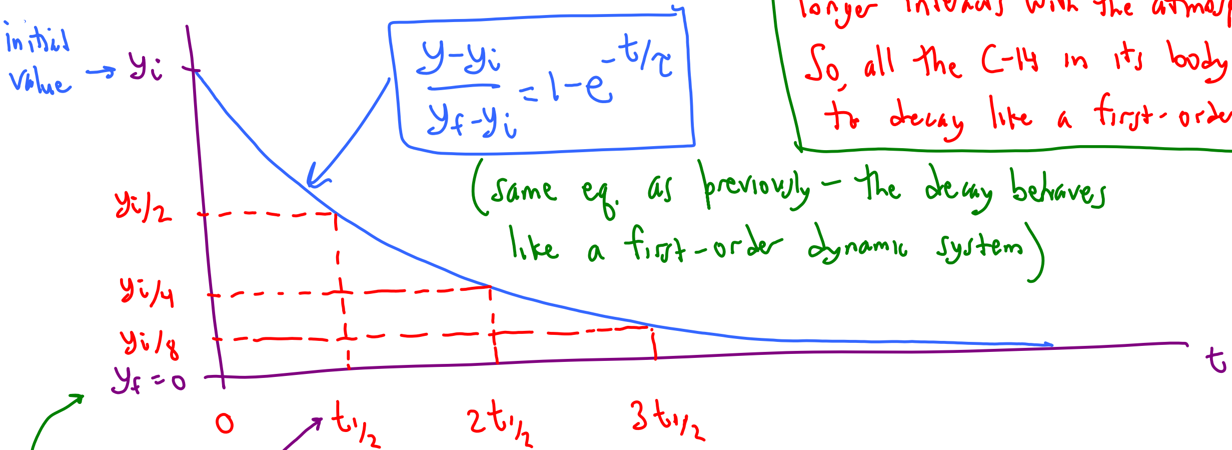
- C-14 decays with a $\boxed{\text{half-life}}$ $\boxed{t_{1/2} = 5730 \text{ years}}$

- Let y = the mol fraction of C-14

At $t=0$, the animal dies and no longer interacts with the atmosphere. So, all the C-14 in its body begins to decay like a first-order system

$$\boxed{\frac{y-y_i}{y_f-y_i} = 1-e^{-t/\tau}}$$

(same eq. as previously – the decay behaves like a first-order dynamic system)



[$y_f = 0$ for C-14 decay because <u>all</u> of it eventually turns into N-14]

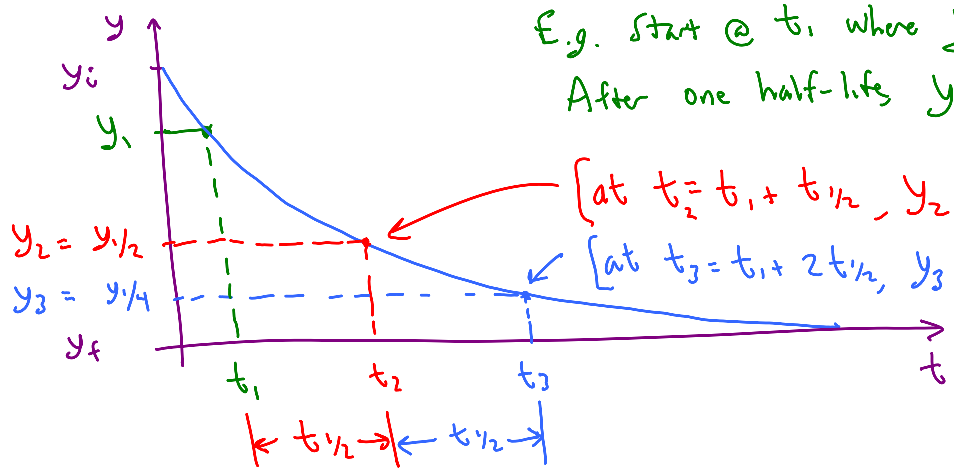
- At $t = t_{1/2}$ (one half-life), <u>half</u> of the C-14 disappears (turns to N-14)
- At $t = 2t_{1/2}$ (two half-lifes), <u>another</u> half of the remaining C-14 disappears
- At $t = 3t_{1/2}$ (3 half-lifes), yet <u>another</u> half of the remaining C-14 disappears

  ⋮ etc. forever

· **Comment**: The half-life definition works no matter when you start

E.g. start @ t_1 where $y = y_1$

After one half-life $y =$ half of y_1



[at $t_2 = t_1 + t_{1/2}$, $y_2 = \frac{1}{2}$ of $y_1 = y_1/2$]

[at $t_3 = t_1 + 2t_{1/2}$, $y_3 = \frac{1}{4}$ of $y_1 = y_1/4$]

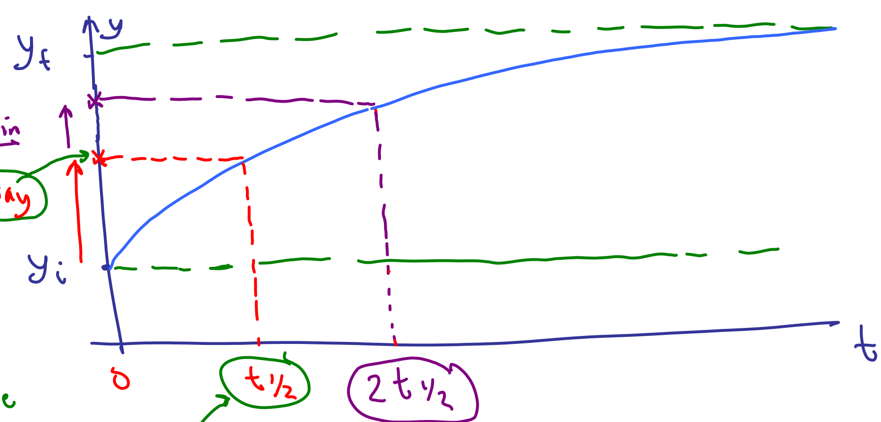
⋮ etc forever

· Applied to measurements, The half-life concept still works for any values of y_i & y_f with a step function input, regardless of whether $y_i > y_f$ or $y_i < y_f$

· E.g., general case

Half-way again

½ way

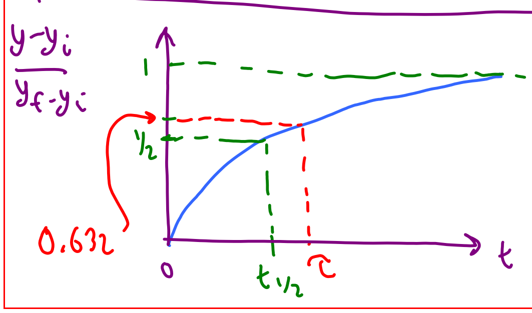


$y = \frac{1}{2}(y_i + y_f)$

[Half way between the initial & final values] when $t =$ one half-life $= t_{1/2}$

· Then, if we think of the present value of y as a new "initial" value, if we wait another half life, y will again change half way from the present value to the final value, ... etc. forever.

Relation between $t_{1/2}$ (half-life) & τ (1st-order time constant):



$t_{1/2}$ is smaller than τ

Set $t = t_{1/2}$ when $\frac{y-y_i}{y_f-y_i} = \frac{1}{2}$

@ $t_{1/2}$, $\frac{y-y_i}{y_f-y_i} = \frac{1}{2} = 1 - e^{-t/\tau}$

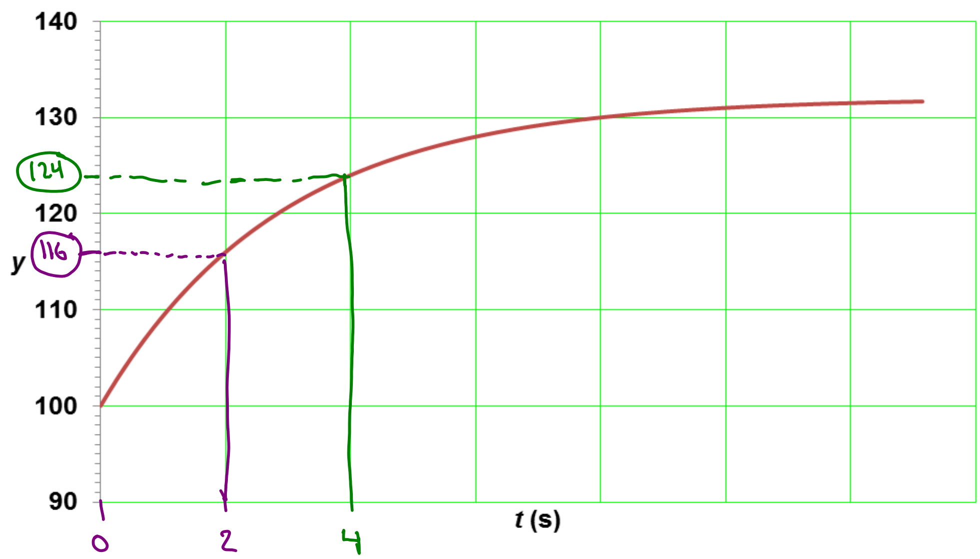
Solve for $t_{1/2}$ → $\boxed{t_{1/2} = -\tau \ln\left(\frac{1}{2}\right) = 0.6931\,\tau}$ ☆

Example: Dynamic system response (first-order and half-life)

Given: Output variable y responds like a first-order dynamic system when exposed to a sudden change of input variable x. The half-life of the system is **2.00 s**. When x is suddenly increased, y grows from an initial value of **100** to a final value of **132**.

To do: Calculate the time (in seconds) required for y to reach a value of **124**.

Solution:



- The total increase in y is from 100 to 132 → $\Delta y_{total} = 32$
- In one half-life, y changes by half of this, $\Delta y_{1/2\,life} = 16$
- So, $y = 100 + 16 = 116$ after one half-life = 2 seconds
- Repeat for another half-life → after another 2 seconds, y starts at $y = 116$ & the final value is $y_f = 132$ → $\Delta y_{total} = 16$

  So, @ $t = 2 + t_{1/2} = 2 + 2 = 4$ s, y = half-way between 116 & 132 = 124

Answer → 2 half lifes = $\boxed{4\text{ s}}$

OR, solve mathematically:

$$\left[\tau = \frac{-t_{1/2}}{\ln(1/2)} = \frac{-2\text{ s}}{\ln(1/2)} = -2.88539\text{ s}\right]$$

$$\frac{y - y_i}{y_f - y_i} = 1 - e^{-t/\tau} \rightarrow \frac{124 - 100}{132 - 100} = 0.75 = 1 - e^{-t/\tau} \rightarrow \text{Solve for } t: \quad t = -\tau \ln(0.25)$$

$$\boxed{t = 4.00\text{ s}}$$ ✓

Example: Dynamic system response (second-order)

[Review the learning module first, for second-order dynamic systems]

Given: The following second-order ODE:

$$5\frac{d^2y}{dt^2}+\frac{dy}{dt}+1000y=x(t) \tag{1}$$

The forcing function is a step function (sudden jump):

$x(t) = 0$ for $t < 0$

$x(t) = 25$ for $t > 0$

(***a***) **To do**: Calculate the natural frequency and damping ratio of this system.

(***b***) **To do**: Calculate the ***equilibrium response*** (as $t \to \infty$, what is y?).

Solution:

(a) Compare to standard form: (first divide Eq. (1) by 1000)

Eq. (1) $$\frac{5}{1000}\frac{d^2y}{dt^2}+\frac{1}{1000}\frac{dy}{dt}+y=\frac{1}{1000}x(t)$$

Standard form $$\frac{1}{\omega_n^2}\frac{d^2y}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dy}{dt}+y=K\,x(t)$$

$K = \frac{1}{1000}$ = Static Sensitivity

$\omega_n^2 = \frac{1000}{5} = 200$

$\omega_n = 14.14 \ \frac{\text{rad}}{\text{s}}$

[ω_n = undamped (radian) natural frequency]

$\frac{2\zeta}{\omega_n} = \frac{1}{1000}$

$\zeta = \frac{\omega_n}{2000} = \frac{14.14}{2000} \rightarrow \boxed{\zeta = 0.00707}$

[ζ = damping ratio]

☆ THIS IS A VERY UNDERDAMPED SECOND-ORDER SYSTEM

(b) Equilibrium response → what is y when $t \to \infty$?

· Answer → Since $y \to$ constant, $\frac{dy}{dt} \to 0$ & $\frac{d^2y}{dt^2} \to 0$, & Eq (1)

becomes $$5\cancel{\frac{d^2y}{dt^2}}_{0}+\cancel{\frac{dy}{dt}}_{0}+1000y=x(t)$$

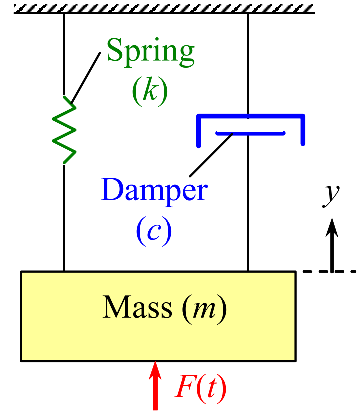
$x = 25$ for all $t > 0$ (this is our given step function)

$1000y = 25 \rightarrow \boxed{y_{\text{equilibrium response}} = 0.025}$ as $t \to \infty$

OR, use $y_{eq.resp} = K X_f$

$= \frac{1}{1000}(25) = 0.025$ ✓

Example: Dynamic system response



Given: A spring-mass-damper system is set up with the following properties: mass m = 22.8 g, spring constant k = 51.6 N/cm, and damping coefficient c = 3.49 N·s/m (c is also called λ in some textbooks). The forcing function is a step function (sudden jump).

To do:

(***a***) Calculate the damping ratio of this system. Will it oscillate?

(***b***) If the system will oscillate, calculate the oscillation frequency in hertz. [*Note*: Calculate the physical frequency, not the radian frequency.] Compare the actual oscillating frequency to the undamped natural frequency of the system.

Solution:

- See learning module: for a spring-mass-damper system

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F(t)$$

- Divide by k:

$$\frac{m}{k}\frac{d^2y}{dt^2} + \frac{c}{k}\frac{dy}{dt} + y = \frac{1}{k}F(t)$$

Careful: ← small k, large K

- Compare to standard form:

$$\frac{1}{\omega_n^2}\frac{d^2y}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dy}{dt} + y = Kx(t)$$

$\boxed{\omega_n = \sqrt{\frac{k}{m}}}$ = undamped natural frequency

$\boxed{\zeta = \frac{\omega_n c}{k} = \frac{c}{2\sqrt{km}}}$ = damping ratio

$\boxed{K = \frac{1}{k}}$ = static sensitivity

$\boxed{x(t) = F(t)}$ = forcing function

(a)
$$\zeta = \frac{3.49\ \text{N·s/m}}{2\sqrt{\left(51.6\ \frac{\text{N}}{\text{cm}}\right)\left(0.0228\ \text{kg}\right)\left(\frac{\text{N·s}^2}{\text{kg·m}}\right)\left(\frac{100\ \text{cm}}{\text{m}}\right)}} = 0.160885 \rightarrow \boxed{\zeta = 0.161}$$

UNDERDAMPED — SYSTEM WILL OSCILLATE ☆

(b)
$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{5160\ \text{N/m}}{0.0228\ \text{kg}}\left(\frac{\text{kg·m}}{\text{s}^2\text{·N}}\right)} = 475.727\ \frac{\text{rad}}{\text{s}} \rightarrow f_n = \frac{\omega_n}{2\pi} = 75.71424\ \text{Hz}$$

$\boxed{\text{Undamped natural frequency} = f_n = 75.7\ \text{Hz}}$

$\boxed{\text{Damped (actual) natural frequency} = f_d = f_n\sqrt{1-\zeta^2} = 74.7\ \text{Hz}}$

☆ NOTICE → DAMPED NATURAL FREQUENCY IS LESS THAN UNDAMPED NAT. FREQ.