Today, we will:

- Finish the pdf module: **Dynamic System Response (response time, log-decrement)**
- Do some example problems – 2nd-order dynamic systems

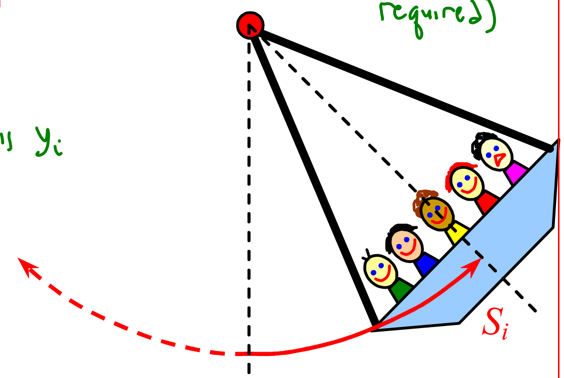
Example: Dynamic system response [*explicit* solution]

Explicit is easy (no iteration required)

Given: A pendulum-type amusement park ride behaves as a 2nd-order dynamic system with damping ratio $\zeta = 0.1$ and $f_n = 0.125$ Hz.

To do: For an initial displacement $S_i = 10.0$ m, calculate the damped natural frequency, the undamped natural frequency, and how long it takes for the oscillations to damp out to within 5% of S_i (the 95% response time).

call this y_i



Solution:

Since we know ζ and f_n, we can plot y or y_{norm} as functions of t or $\omega_n t$, using the equation for underdamped 2nd-order dynamic system response, as given in the learning module,

$$y_{\text{norm}} = \frac{y - y_i}{y_f - y_i} = 1 - e^{-\zeta\omega_n t}\left[\frac{1}{\sqrt{1-\zeta^2}}\sin\left(\omega_n t\sqrt{1-\zeta^2} + \sin^{-1}\left(\sqrt{1-\zeta^2}\right)\right)\right]$$

Eq. (1)

Solve for y to plot y vs. t → $\boxed{y = y_{\text{norm}}(y_f - y_i) + y_i}$

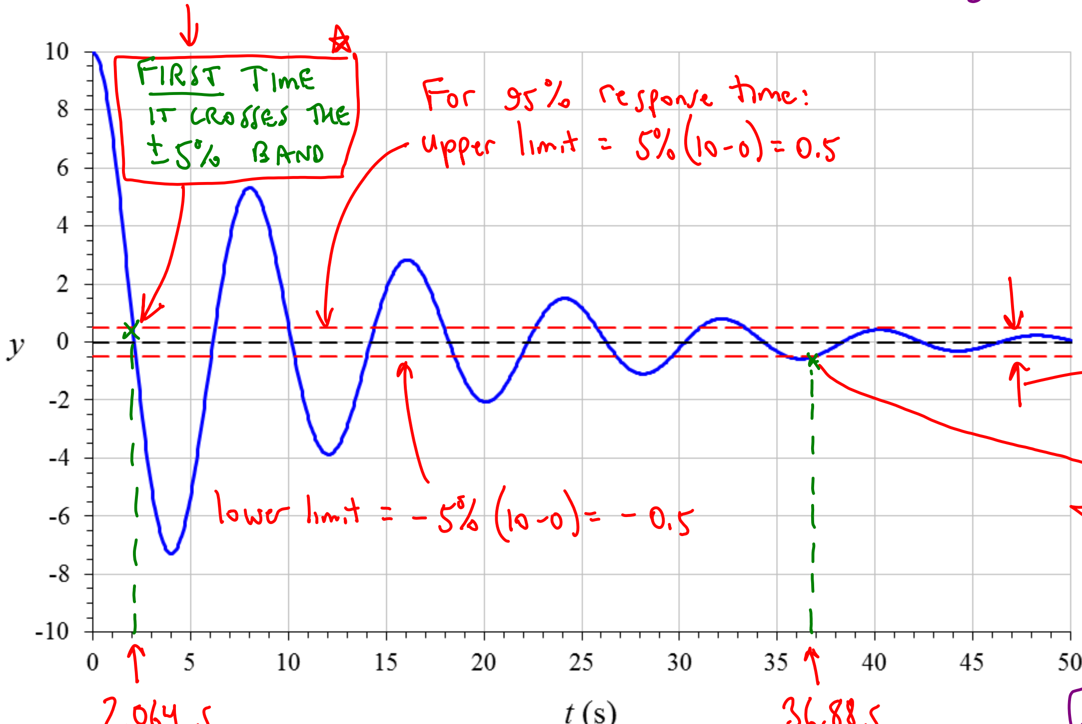
Numbers: $y_i = 10$ m

$y_f = 0$ m (will stop eventually)

$\zeta = 0.1$

$f_n = 0.125$ Hz

$\omega_n = 2\pi f_n = 0.785398 \frac{\text{rad}}{\text{s}}$



THIS POINT IS THE LAST TIME IT CROSSES INTO THE ±5% BAND

$\boxed{f_n = 0.125 \text{ Hz}}$

$f_d = f_n\sqrt{1-\zeta^2}$

$\boxed{f_d = 0.124 \text{ Hz}}$

ANSWERS:

$t_{95\%}$ rise time = 2.06 s

$t_{95\%}$ response time (settling time) = 36.9 s

Example: Dynamic system response [*implicit* solution]

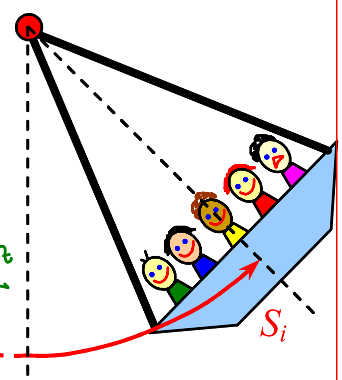
must iterate

Given: A pendulum-type amusement park ride behaves as a 2nd-order dynamic system. We measure the *actual* (damped) period of the oscillations, T_d = 7.65 s. We also measure the two peak amplitudes at 4 periods apart, namely, S_i = 4.740 m at the first observed peak, when t = 7.65 s, and S_i = 0.239 m at a peak four periods later, when t = 38.25 s.

$f_d = \frac{1}{T_d}$

$f_d = 0.1307$ Hz

$\omega_d = 2\pi f_d = 0.8213$ rad/s

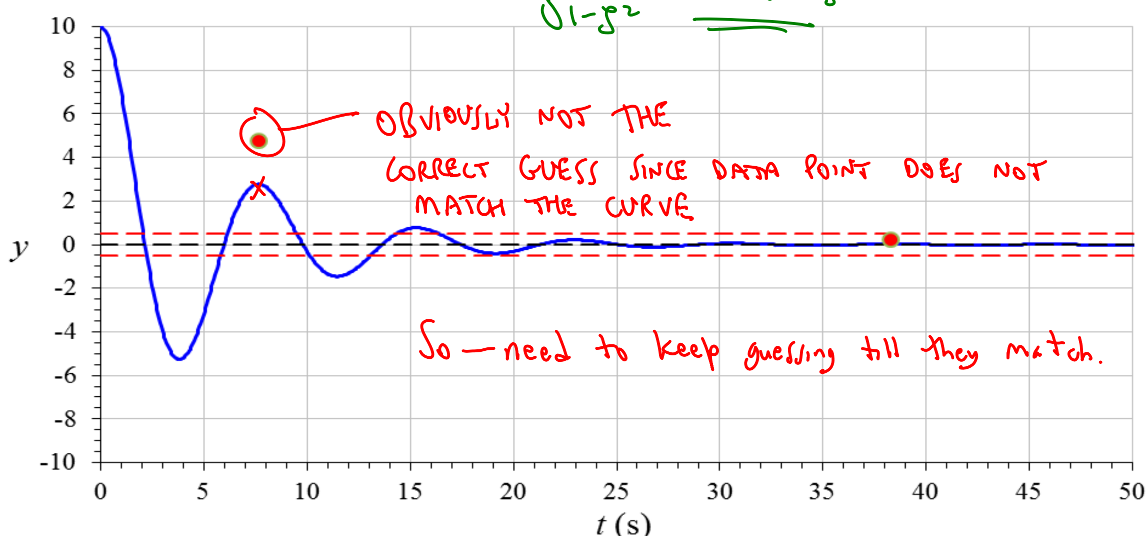


To do: Calculate the damping ratio and the undamped natural frequency.

Solution:

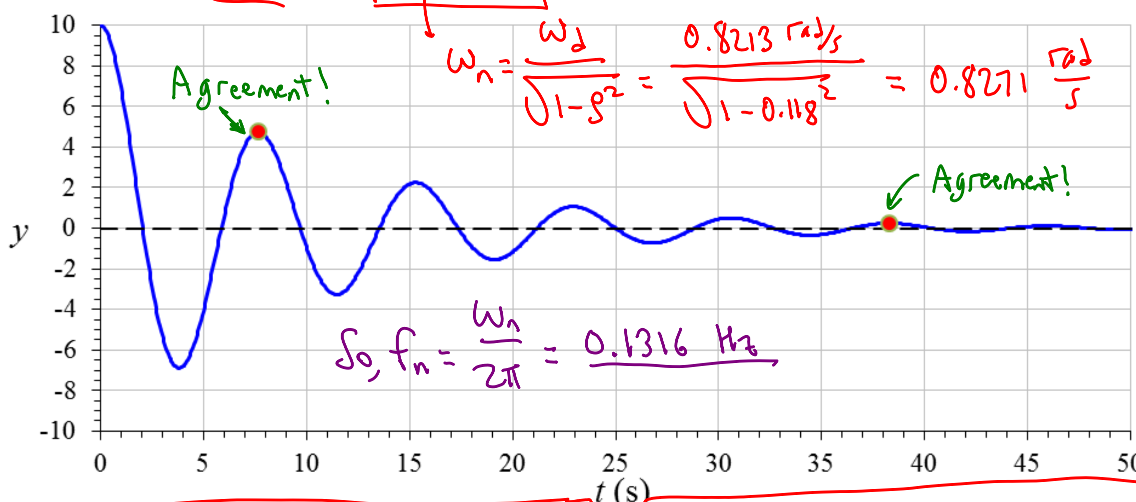
Here we do *not* know ζ or ω_n, so this is an implicit solution, involving iteration. To solve graphically, we *guess* ζ, then plot using the equation for underdamped 2nd-order dynamic system response, as given in the learning module. We iterate until we satisfy both of the observed data points. Two cases are shown:

- Initial guess is ζ = 0.200. → $\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = 0.8382 \frac{\text{rad}}{\text{s}}$



- Iterate until converge on $\boxed{\zeta = \mathbf{0.118}}$. SEE EXCEL FILE POSTED ON WEBSITE

$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \frac{0.8213 \text{ rad/s}}{\sqrt{1-0.118^2}} = 0.8271 \frac{\text{rad}}{\text{s}}$



ANSWERS: $\boxed{\zeta = 0.118 = \text{damping ratio}}$ $\boxed{f_n = 0.132 \text{ Hz} = \text{Undamped natural frequency}}$

SIMPLIFIED METHODS TO DETERMINE ζ (DAMPING RATIO):

(without having to plot the curve)

1. MAGNITUDE-ONLY APPROACH → Ignore the sine function in the equation for y_{normal} since $\sin(\sim) = 1$ at the peaks

- Take absolute value of the y_{norm} equation (for $\zeta < 1$)

$$|y_{norm}| = \left|\frac{y-y_i}{y_f-y_i}\right| = 1 - e^{-\zeta\omega_n t} \cdot \frac{1}{\sqrt{1-\zeta^2}} \boxed{|\sin(\sim)|}$$

[$\sin(\sim)$ ranges from -1 to 1, so at either peak (max or min), $|\sin(\sim)| = 1$] → Set this magnitude = 1 since we are interested only in the peaks

- Knowns: $\omega_d = 0.8213$ rad/s (damped natural radian frequency)
  $y_f = 0$ (final response or equilibrium response)
- Unknowns: y_i, ζ, ω_n → So, we need 3 equations

(1) $\omega_n = \dfrac{\omega_d}{\sqrt{1-\zeta^2}}$

(2) At the first peak, $y_1 = 4.740$ m @ $t_1 = 7.65$ s

$$\therefore y_{norm_1} = \boxed{\frac{y_1-y_i}{y_f-y_i} = 1 - e^{-\zeta\omega_n t_1}\left(\frac{1}{\sqrt{1-\zeta^2}}\right)}$$

(3) At the other peak, $y_2 = 0.239$ m @ $t_2 = 38.25$ s

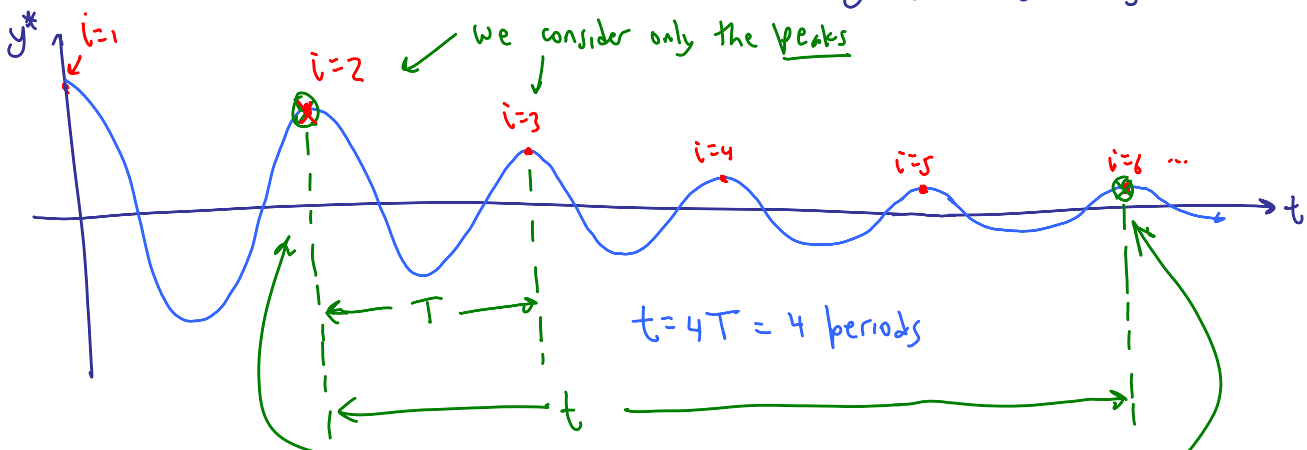
$$\therefore y_{norm_2} = \boxed{\frac{y_2-y_i}{y_f-y_i} = 1 - e^{-\zeta\omega_n t_2}\left(\frac{1}{\sqrt{1-\zeta^2}}\right)}$$

- Solve these 3 simultaneous eq's any way you can (trial & error, Matlab, Excel, EES ...)

Answers: $\boxed{\zeta = 0.118}$ $\boxed{y_i = 10.003 \text{ m}}$ $\boxed{\omega_n = 0.82708 \ \frac{\text{rad}}{\text{s}}}$ → $\boxed{f_n = \frac{\omega_n}{2\pi} = 0.1316 \text{ Hz}}$

2. THE LOG-DECREMENT METHOD (see learning module for details)

• Let $y^* = 1 - y_{norm}$ = oscillatory part only (oscillates around $y^* = 0$ & asymptotes to $y^* = 0$ as $t \to \infty$)



Our two data points are at i=2 & i=6

• In the log-decrement method (see learning module), n = # periods between the two data pts = 4

$\boxed{n=4}$ $\left[y_{i=2} = y_2 \quad \text{\&} \quad y_{i+n} = y_{2+4} = y_6 \right]$

• Calculate y^* values: @ i=2, $y^*_2 = 1 - y_{norm} = 1 - \dfrac{y_2 - y_i}{y_f - y_i} = 1 - \dfrac{4.740 - 10}{0 - 10} = \underline{\underline{0.474}}$

@ i=6, $y^*_6 = 1 - y_{norm} = 1 - \dfrac{y_6 - y_i}{y_f - y_i} = 1 - \dfrac{0.239 - 10}{0 - 10} = \underline{\underline{0.0239}}$

• Use equations from the learning module:

$$\delta = \text{log decrement} = \frac{\ln\left(\dfrac{y^*_i}{y^*_{i+n}}\right)}{n} = \frac{\ln\left(\dfrac{y^*_2}{y^*_6}\right)}{4} = \frac{\ln(0.474/0.0239)}{4} = \underline{\underline{0.7468 = \delta}}$$

$$\zeta = \text{damping ratio} = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = \frac{0.7468}{\sqrt{(2\pi)^2 + (0.7468)^2}} = 0.11803$$

(Same answer as with the other two methods) → $\boxed{\zeta = 0.118}$

• From this ζ, we calculate $f_n = \dfrac{f_d}{\sqrt{1-\zeta^2}} = 0.1316 \approx \boxed{0.132 \text{ Hz} = f_d}$

Example: Dynamic system response (second-order, log-decrement method)

Given: Output variable y responds like a first-order dynamic system when exposed to a sudden change of input variable x. Bill measures the following (oscillatory components) at two peaks, 5 peaks apart from each other:

- $y^*_1 = 1.00$ at $t_1 = 0.00$ s.
- $y^*_6 = 0.200$ at $t_6 = 0.250$ s.

To do: Use the log-decrement method to calculate the damping ratio of this system.

Solution: Here are the log-decrement equations for your convenience:

$$\ln\left(\frac{y^*_i}{y^*_{i+n}}\right) = n\delta \qquad \zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} \qquad \omega_d = \frac{2\pi}{T} = 2\pi f_d \qquad \omega_n = 2\pi f_n = \frac{\omega_d}{\sqrt{1-\zeta^2}}.$$

• Pt 1 is @ $i=1$ & pt 2 is @ $i=6$ ∴ $n = 6-1 = 5$ $\boxed{n=5}$

$$\delta = \frac{\ln(y^*_1 / y^*_6)}{n} = \frac{\ln(1.00/0.200)}{5} = \underline{\underline{0.321888}} = \delta = \text{log decrement}$$

$$\zeta = \text{damping ratio} = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = \frac{0.321888}{\sqrt{(2\pi)^2 + (0.32188)^2}} = 0.0511629$$

or $\boxed{\zeta \approx 0.0512}$