Outline and Equation Sheet for M E 345

Author: John M. Cimbala, Penn State University
Latest revision, 16 December 2014

Introduction

- Primary dimensions – mass, length, time, temperature, current, amount of light, and amount of matter.
- Significant digits – the rules for multiplication and division, and for addition and subtraction.
- Rounding off – round *up* if the least significant digit is *odd*, and *truncate* if the least significant digit is *even*.

Dimensional Analysis

- Law of dimensional homogeneity – *Every additive term in an equation must have the same dimensions*.
- The method of repeating variables – there are 6 steps:
  1. List the parameters and count them, n.
  2. List the primary dimensions of each parameter.
  3. Set the reduction, j, as the number of primary dimensions in the problem. Then $k = n - j$. [Reduce j by one if necessary.]
  4. Choose j repeating variables.
  5. Construct the k Πs, and manipulate as necessary.
  6. Write the final functional relationship $\Pi_1 = \text{function}(\Pi_2, \Pi_3, \ldots \Pi_k)$ and check your algebra.
- Dimensional analysis is often extremely useful in setting up and designing experiments.

Errors and Calibration

- Systematic errors (bias errors) – consistent, repeatable errors.
- Random errors (precision errors) – scatter in data, a lack of repeatability, unrepeatable, inconsistent errors.
- Accuracy – *accuracy error is the measured value minus the true value*.
- Precision – *precision error is the reading minus the average of readings*.
- Other errors – zero, linearity, sensitivity, resolution, hysteresis, instrument reapeatability, drift.
- Calibration – static (time not relevant) vs. dynamic (time is relevant) calibration.
- Mean bias error – defined as $\text{MBE} = \frac{1}{n}\sum_{i=1}^{n}\frac{x_i - x_{\text{true}}}{x_{\text{true}}}$, and usually expressed as a percentage.

Basic Electronics Review

- Unity conversion factors – $\left(\frac{1\ \Omega}{1\ \text{V/A}}\right) = 1$, $\left(\frac{1\ \text{F}}{1\ \text{C/V}}\right) = 1$, $\left(\frac{1\ \text{H}\cdot\text{A/s}}{1\ \text{V}}\right) = 1$, and $\left(\frac{1\ \text{A}\cdot\text{s}}{1\ \text{C}}\right) = 1$.
- Ohm's law and basic equations – $\Delta V = IR$, $I = C\frac{dV}{dt}$ and $V = L\frac{dI}{dt}$ at any instant in time.
- Voltage divider – $V_{\text{out}} = V_{\text{in}}\frac{R_2}{R_1 + R_2}$ where V_{out} is measured across resistor R_2.
- Resistors in series – $R_{\text{total}} = R_1 + R_2 + \ldots + R_N$ and $I = I_1 = I_2 = \ldots = I_N$.
- Resistors in parallel – $R_{\text{total}} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_N}}$ and $I_{\text{total}} = I_1 + I_2 + \ldots + I_N$.
- Capacitors in series – $C_{\text{total}} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_N}}$ and $I = I_1 = I_2 = \ldots = I_N$.
- Capacitors in parallel – $C_{\text{total}} = C_1 + C_2 + \ldots + C_N$ and $I_{\text{total}} = I_1 + I_2 + \ldots + I_N$.
- Inductors in series – $L_{\text{total}} = L_1 + L_2 + \ldots + L_N$ and $I = I_1 = I_2 = \ldots = I_N$.
- Inductors in parallel – $L_{\text{total}} = \frac{1}{\frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_N}}$ and $I_{\text{total}} = I_1 + I_2 + \ldots + I_N$.
- Impedance – $|\mathbf{Z}| = \frac{V_{\text{peak}}}{I_{\text{peak}}}$: resistor $\mathbf{Z} = R$, capacitor $\mathbf{Z} = \frac{1}{i\omega C}$, inductor $\mathbf{Z} = i\omega L$.

Basic Statistics

- Definitions – sample mean $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, sample standard deviation $S = \sqrt{\frac{\sum_{i=1}^{n} d_i^2}{n-1}} = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}}$, sample variance ($S^2$), sample median (half lower, half higher), sample mode (most probable value – one that occurs most frequently), population (all values) vs. sample (a selected portion of the total population).
- Excel – learn how to use Excel's built-in statistics functions.
- Root mean square error – defined as $\text{RMSE} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i - x_{\text{true}}}{x_{\text{true}}}\right)^2}$, and usually expressed as a percentage.

Histograms and Probability Density Functions

- Histogram plots – frequency, bins or classes, bin width or class width, Sturgis and Rice rules.
- Normalized histograms – how to normalize the vertical scale and the horizontal scale.
- PDF – how to create a probability density function from a histogram.
- Expected value – same as population mean.
- Standard deviation – same as population standard deviation.
- Normalized PDF – transform from x to z using $z = \frac{x-\mu}{\sigma}$ and $f(z) = \sigma f(x)$.
- The Gaussian or normal PDF – how to predict probabilities for systems with purely random errors using the error function, $A(z) = \frac{1}{2}\text{erf}\left(\frac{z}{\sqrt{2}}\right)$ where $\text{erf}(\eta) = \frac{2}{\sqrt{\pi}}\int_{\xi=0}^{\xi=\eta}\exp(-\xi^2)\,d\xi$. See table of $A(z)$.
- Other PDFs – lognormal, chi-squared distribution, student's t, degrees of freedom, confidence level and significance level.
- How to estimate the population mean μ from a sample using the student's t PDF: $\mu = \overline{x} \pm t_{\alpha/2}\frac{S}{\sqrt{n}}$ with confidence level $1-\alpha$; use table of critical values for the student's t PDF.
- How to estimate the population standard deviation σ from a sample using the χ^2 PDF: $(n-1)\frac{S^2}{\chi^2_{\alpha/2}} \le \sigma^2 \le (n-1)\frac{S^2}{\chi^2_{1-\alpha/2}}$ with confidence level $1-\alpha$; use table of critical values for the χ^2 PDF.
- The central limit theorem (CLT) – standard error of the mean = standard deviation of sample means = $S_{\overline{x}} = \frac{S_{\text{overall}}}{\sqrt{n}}$. As n gets large, we write the CLT in terms of population standard deviations: $\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}}$.

Correlation and Regression

- Linear correlation coefficient – $r_{xy} = \frac{\sum_{i=1}^{i=n}(x_i-\overline{x})(y_i-\overline{y})}{\sqrt{\sum_{i=1}^{i=n}(x_i-\overline{x})^2\sum_{i=1}^{i=n}(y_i-\overline{y})^2}}$. By definition, r_{xy} must always lie between −1 and 1, i.e., $-1 \le r_{xy} \le 1$. To determine if there is a trend in the data, use the table of critical values of r_{xy}.
- Regression analysis – least-squares fits (linear, polynomial, and multiple variables), standard error of estimate. Equations are provided in the lecture notes, but we typically use Excel to do the analysis.

  Standard error of estimate – also called ***standard error***: $S_{y,x} = \sqrt{\frac{\sum_{i=1}^{i=n} y_i^2 - b\sum_{i=1}^{i=n} y_i - a\sum_{i=1}^{i=n} x_i y_i}{n-2}}$.

Outlier Points

- Outliers in a set of data points – use the modified Thompson tau technique: compare maximum absolute value of the deviation with Thompson τ times sample standard deviation (τS) to determine outliers: If $\delta > \tau S$, reject the outlier.
- Outliers in a set of data pairs – for polynomial fit of order m, $S_{y,x} = \sqrt{\dfrac{\sum_{i=1}^{i=n}(y_i - Y_i)^2}{\text{df}}}$, $\text{df} = n - (m+1)$.

  Examine the standardized residual $e_i / S_{y,x}$ to determine outliers. There are ***two criteria*** for a data pair to be called an outlier:
  - $\left|e_i / S_{y,x}\right| > 2$
  - The standardized residual is *inconsistent* with its neighbors (as judged by a plot of $e_i / S_{y,x}$ vs. x).

Experimental Uncertainty Analysis

- Maximum uncertainty – assumes all errors have the same sign and add up (unlikely). Normally, we do not use the maximum uncertainty. Typically, we use instead the RSS uncertainty, as described below:
- RSS uncertainty or expected uncertainty – assumes some errors are positive and some are negative (more likely). Use $u_{R,\text{RSS}} = \sqrt{\sum_{i=1}^{i=N}\left(u_{x_i}\dfrac{\partial R}{\partial x_i}\right)^2}$ or, if $R = C \cdot x_1^{a_1} \cdot x_2^{a_2} \ldots \cdot x_N^{a_N}$, $\dfrac{u_R}{R} = \dfrac{u_{R,\text{RSS}}}{R} = \sqrt{\sum_{i=1}^{i=N}\left(a_i \dfrac{u_{x_i}}{x_i}\right)^2}$.
- Combining elemental uncertainties using RSS uncertainty analysis – use $u_x = \sqrt{\sum_{i=1}^{i=K} u_i^2}$, where K is the number of elemental uncertainties, and u_i are the individual elemental uncertainties.

Experimental Design

- Choosing a test matrix – one parameter at a time (full factorial) vs. Taguchi's experimental design arrays (fractional factorial) test matrices. $N = L^P$ (L = levels, P = parameters) for a full factorial experiment.
- Level averages – E.g., average over all the runs where parameter a is at level 1.
- Optimum Taguchi design array – Meets two criteria:
  - *Each level of each parameter appears the same number of times in the array.*
  - *Repetitions of parameter-level combinations are minimized as much as possible.*
- Taguchi orthogonal arrays – know how to spot a poorly designed experimental design array.
- Response surface methodology (RSM) – goal is to efficiently hunt for the optimum values of parameters a, b, c, etc. such that response y is maximized or minimized.
- Coded variables – transform the physical variables into coded variables, $x_1 = 2\left(\dfrac{a - a_{\text{mid value}}}{a_{\text{range}}}\right)$, $a_{\text{mid value}} = \dfrac{a_{\min} + a_{\max}}{2}$, $a_{\text{range}} = a_{\max} - a_{\min}$ (with similar equations for the other variables). This is necessary to properly use the RSM technique.
- Direction of steepest ascent – vector $\left(\dfrac{\partial y}{\partial x_1}, \dfrac{\partial y}{\partial x_2}, \dfrac{\partial y}{\partial x_3}\right)$ determined by regression analysis on coded variables.

  For known Δx_1, $\Delta x_2 = \Delta x_1 \dfrac{\partial y}{\partial x_2}\Big/\dfrac{\partial y}{\partial x_1}$ and $\Delta x_3 = \Delta x_1 \dfrac{\partial y}{\partial x_3}\Big/\dfrac{\partial y}{\partial x_1}$.
- Marching – Transform back to physical variables, $\Delta a = \dfrac{a_{\text{range}}}{2}\Delta x_1$, then march in the direction of steepest ascent until y is no longer improving.

Hypothesis Testing

- Null hypothesis – *a theory that is being considered or tested*. Typically, the null hypothesis represents "nothing is happening."
- There are two parts to the null hypothesis:
  - The critical value (extreme value), μ_0.
  - The "sides" or "tails" of the null hypothesis – select the least likely scenario ($\mu = \mu_0$, $\mu < \mu_0$, or $\mu > \mu_0$).
- Alternative hypothesis – also called the research hypothesis – *the complement of the null hypothesis*.
- *t*-statistic – calculate for the extreme value (critical value) of the null hypothesis, $t = \dfrac{\bar{x} - \mu_0}{S/\sqrt{n}}$.
- *p*-value – *p*-value is *the probability of wrongly rejecting the null hypothesis, if it is in fact true*. In Excel, use **TDIST(*t*, df, tails)** (*p*-value is the area under the tail(s) of the *t*-PDF). See also the tables of *p*-values for a given value of df. *Note*: The values in the table are for ***one-tail*** hypothesis tests; multiply by 2 for two tails.
- A ***small*** *p*-value means that the null hypothesis is ***unlikely to be true*** – we reject the null hypothesis if the *p*-value is less than some level, typically 0.05 (5%) to standard engineering confidence level.
- Paired samples hypothesis testing when *n* is the same for the two samples – use $\delta = x_B - x_A$ as the variable, set the null hypothesis as $\mu_0 = \mu_\delta = 0$, and set the value of the critical *t*-statistic to $t = \dfrac{\bar{\delta} - \mu_0}{S_\delta/\sqrt{n}}$.
- Two sample hypothesis testing when *n* is *not* the same for the two samples – set the null hypothesis as $\mu_A = \mu_B$, set the critical *t*-statistic to $t = \dfrac{\bar{x}_A - \bar{x}_B}{\sqrt{\dfrac{S_A^2}{n_A} + \dfrac{S_B^2}{n_B}}}$, and use $\text{df} = \text{NINT}\left[\dfrac{\left(\dfrac{S_A^2}{n_A} + \dfrac{S_B^2}{n_B}\right)^2}{\dfrac{1}{n_A - 1}\left(\dfrac{S_A^2}{n_A}\right)^2 + \dfrac{1}{n_B - 1}\left(\dfrac{S_B^2}{n_B}\right)^2}\right]$ (Welch's equation) as a kind of weighted value of the degrees of freedom.

Digital Data Acquisition

- Binary to decimal and vice-versa – how to convert: sum columns or successive division, respectively.
- A/D systems – $\text{resolution} = \dfrac{(V_{\max} - V_{\min})}{2^N}$, $\text{quantization error} = \pm\dfrac{1}{2}\text{resolution} = \pm\dfrac{1}{2}\dfrac{(V_{\max} - V_{\min})}{2^N}$.
- Discrete sampling – $\Delta t = 1/f_s$, $\omega = 2\pi f$, watch out for clipping and aliasing.
- Ailasing – $\text{If } f_s > 2f, \text{ then there is no aliasing}$ (Nyquist criterion), $\text{if } \dfrac{2}{3}f < f_s < 2f, \text{ then } f_a = |f_s - f|$. $\text{If } f_s < \dfrac{2}{3}f, \text{ then } f_a = \left(\dfrac{f_a}{f_{\text{folding}}}\right) f_{\text{folding}}$ where $f_{folding} = \dfrac{f_s}{2}$, and need to use the folding diagram.

  OR, use the general equation $f_{\text{perceived}} = \left| f - f_s \cdot \text{NINT}\left(\dfrac{f}{f_s}\right)\right|$, where NINT is the "nearest integer".
- Signal reconstruction – the Cardinal series: reconstruct a digital signal back into an analog signal.
- Fourier series – $f(t) = c_0 + \sum_{n=1}^{\infty} a_n \sin(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \cos(n\omega_0 t)$ where fundamental frequency is $f_0 = 1/T$ and $\omega_0 = 2\pi f_0$. Coefficients are $c_0 = \dfrac{1}{T}\int_0^T f(t)\,dt$, $a_n = \dfrac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)\,dt$, and $b_n = \dfrac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)\,dt$.
- Harmonic amplitude plots – plot of amplitude of each coefficient $|a_1|$, $|a_2|$, etc. versus harmonic number *n*.
- Fourier transforms, DFTs and FFTs – discrete version of Fourier series; $f_{\max} = f_{\text{folding}} = \dfrac{f_s}{2} = \dfrac{N}{2}\Delta f$ where N = total number of discrete data points taken, T = total sampling time, $\Delta t = T/N$ = time between data points, $f_s = 1/\Delta t = N/T$ = sampling frequency, and $\Delta f = 1/T$ = frequency resolution.
- Leakage – appears when the discrete data acquisition does not stop at exactly the same phase as it started.
- Windowing – reduces but does not totally eliminate leakage.
- Anti-aliasing filter – use a low-pass filter to remove high frequencies to avoid aliasing.

Filters

- 1st-order low-pass filter – $f_{\text{cutoff}} = \dfrac{\omega_{\text{cutoff}}}{2\pi} = \dfrac{1}{2\pi RC}$, $G = \dfrac{|V_{\text{out}}|}{|V_{\text{in}}|} = \dfrac{1}{\sqrt{1+(f/f_{\text{cutoff}})^2}}$, $\phi = -\arctan\left(\dfrac{f}{f_{\text{cutoff}}}\right)$.
- 1st-order high-pass filter – $f_{\text{cutoff}} = \dfrac{\omega_{\text{cutoff}}}{2\pi} = \dfrac{1}{2\pi RC}$, $G = \dfrac{|V_{\text{out}}|}{|V_{\text{in}}|} = \dfrac{1}{\sqrt{1+(f_{\text{cutoff}}/f)^2}}$, $\phi = \arctan\left(\dfrac{f_{\text{cutoff}}}{f}\right)$.
- Higher-order filters – for a Butterworth low-pass filter of order n, $G = \dfrac{|V_{\text{out}}|}{|V_{\text{in}}|} = \dfrac{1}{\sqrt{1+(f/f_{\text{cutoff}})^{2n}}}$.
- Gain expressed in decibels – for any filter or amplifier, $G_{\text{dB}} = 20\log_{10} G = 20\log_{10}\dfrac{|V_{\text{out}}|}{|V_{\text{in}}|}$.

Operational Amplifiers (op-amps)

- Open-loop gain – $V_{\text{o}} = g(V_p - V_n)$.
- Closed-loop configuration (with feedback loop) – $V_p \approx V_n$.
- Example circuits – buffer: $V_{\text{out}} = V_{\text{in}}$, inverting amplifier: $V_{\text{out}} = -\dfrac{R_2}{R_1}V_{\text{in}} = GV_{\text{in}}$, inverter: $V_{\text{out}} = -V_{\text{in}}$, inverting summer: $V_{\text{out}} = -(V_1 + V_2)$, noninverting amplifier: $V_{\text{out}} = (1+(R_2/R_1))V_{\text{in}} = GV_{\text{in}}$.
- Other circuits – active filters, clipping circuits (high and low voltage clipping), etc.
- Common-mode rejection ratio (CMRR) – $\text{CMRR} = 20\log_{10}\dfrac{g}{G_{\text{CM}}}$, and G_{CM} is the common-mode gain.
- Gain-bandwidth product (GBP) – op-amp acts like low-pass filter at high frequencies:
  - *Noninverting* op-amp amplifier: $\text{GBP} = \text{GBP}_{\text{noninverting}} = f_c G_{\text{theory, noninverting}} = \text{constant}$ [GBP supplied by manufacturer's specs].
  - *Inverting* op-amp amplifier: $\text{GBP}_{\text{inverting}} = \dfrac{-R_2}{R_1+R_2}\text{GBP}_{\text{noninverting}}$, $\text{GBP}_{\text{inverting}} = f_c G_{\text{theory, inverting}} = \text{constant}$.
- Loading – input and output loading can cause voltage change due to internal resistances. R_i = input resistance or input impedance.

Stress, Strain, and Strain Gages

- Axial stress and axial strain – $\sigma_a = \dfrac{F}{A}$, $\varepsilon_a = \dfrac{\delta L}{L}$, $\sigma_a = E\varepsilon_a$.
- Transverse strain & Poisson's ratio – $\nu = \text{Poisson's ratio} = -\dfrac{\varepsilon_t}{\varepsilon_a}$, $\varepsilon_t = \dfrac{\delta w}{w} = \dfrac{\delta t}{t}$; $\varepsilon_t = \dfrac{\delta D}{D}$ for round rods.
- Stress-strain relationship on a surface; principal axes – $\varepsilon_x = \dfrac{1}{E}(\sigma_x - \nu\sigma_y)$, $\varepsilon_y = \dfrac{1}{E}(\sigma_y - \nu\sigma_x)$.
- Wire resistance – $R = \dfrac{\rho L}{A}$, $\dfrac{\delta R}{R} = S\varepsilon_a$, where S is the strain gage factor, defined as $S = \dfrac{\delta R/R}{\varepsilon_a}$.
- Wheatstone bridge – $V_{\text{o}} = V_s\dfrac{R_3R_1 - R_4R_2}{(R_2+R_3)(R_1+R_4)}$, bridge is balanced when $R_3R_1 = R_4R_2$. Approximate relationship when resistances change: $\dfrac{V_{\text{o}}}{V_s} \approx \dfrac{R_{2,\text{initial}}R_{3,\text{initial}}}{(R_{2,\text{initial}}+R_{3,\text{initial}})^2}\left(\dfrac{\delta R_1}{R_{1,\text{initial}}} - \dfrac{\delta R_2}{R_{2,\text{initial}}} + \dfrac{\delta R_3}{R_{3,\text{initial}}} - \dfrac{\delta R_4}{R_{4,\text{initial}}}\right)$.
- Quarter, half, and full bridge – $\varepsilon_a \approx \dfrac{4}{n}\dfrac{V_{\text{o}}}{V_s}\dfrac{1}{S}$ or $V_{\text{o}} \approx \dfrac{n}{4}\varepsilon_a S V_s$, where n = 1 for a quarter bridge. n = 2 for a half bridge, and n = 4 for a full bridge.
- Unbalanced bridge – $\varepsilon_a \approx \dfrac{4}{n}\dfrac{(V_{\text{o}} - V_{\text{o,reference}})}{V_s}\dfrac{1}{S}$ or $V_{\text{o}} \approx V_{\text{o,reference}} + \dfrac{n}{4}\varepsilon_a S V_s$.

Dynamic System Response

- General governing equation (ODE): $a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + ... + a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = bx$.
- First-order system: $\tau \frac{dy}{dt} + y = Kx$, $K = b/a_0$, $\tau = a_1/a_0$.
    - For step-function input, $\frac{y - y_i}{y_f - y_i} = \text{ nondimensionalized output} = 1 - e^{\frac{-t}{\tau}}$, $\Gamma(t) = \frac{y - y_f}{y_i - y_f} = e^{\frac{-t}{\tau}}$.
    - For ramp function input, $y_{\text{measured}} = KA\left[t - \tau\left(1 - e^{-(t/\tau)}\right)\right]$, $\text{time lag} = \tau$, $\text{error} = y_{\text{measured}} - y_{\text{ideal}} = -KA\tau$.
- Second-order system: $\frac{1}{\omega_n^2}\frac{d^2 y}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dy}{dt} + y = Kx$, $K = b/a_0$, $\omega_n = \sqrt{\frac{a_0}{a_2}}$, $\zeta = \frac{a_1}{2\sqrt{a_0 a_2}}$. For step-function:
    - ***Underdamped***, $\zeta < 1$, $\frac{y - y_i}{y_f - y_i} = 1 - e^{-\zeta\omega_n t}\left[\frac{1}{\sqrt{1-\zeta^2}}\sin\left(\omega_n t\sqrt{1-\zeta^2} + \phi\right)\right]$, $\phi = \sin^{-1}\left(\sqrt{1-\zeta^2}\right)$.

      Undamped natural frequency and ringing frequency: $f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{a_0}{a_2}}$, and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ or $f_d = f_n\sqrt{1-\zeta^2}$.
    - ***Critically damped***, $\zeta = 1$, $\frac{y - y_i}{y_f - y_i} = 1 - e^{-\omega_n t}\left(1 + \omega_n t\right)$, and there is no phase shift.
    - ***Overdamped***, $\zeta > 1$, $\frac{y - y_i}{y_f - y_i} = 1 - e^{-\zeta\omega_n t}\left[\cosh\left(\omega_n t\sqrt{\zeta^2 - 1}\right) + \frac{\zeta}{\sqrt{\zeta^2 - 1}}\sinh\left(\omega_n t\sqrt{\zeta^2 - 1}\right)\right]$.
    - ***Log-decrement method***: $\ln\left(\frac{y^*_i}{y^*_{i+n}}\right) = n\delta$, $\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}$, $\omega_d = \frac{2\pi}{T} = 2\pi f_d$, $\omega_n = 2\pi f_n = \frac{\omega_d}{\sqrt{1-\zeta^2}}$.

Temperature Measurement

- Mechanical devices – liquid-in-glass thermometers, bimetallic strips, pressure thermometers.
- Thermojunctive devices – thermocouples, sensing junctions, reference junctions, type K, T, etc.
- Thermocouple laws – notation: $V_{1-2} = V_{1-R} - V_{2-R}$.
    - **Law of intermediate metals**: A third (intermediate) metal wire can be inserted in series with one of the wires without changing the voltage reading (provided that the two new junctions are at the same temperature).
    - **Law of intermediate temperatures**: If identical thermocouples measure the temperature difference between T_1 and T_2, and the temperature difference between T_2 and T_3, then the sum of the corresponding voltages $V_{1-2} + V_{2-3}$ must equal the voltage V_{1-3} generated by an identical thermocouple measuring the temperature difference between T_1 and T_3, i.e., $V_{1-3} = V_{1-2} + V_{2-3}$ for any 3 temperatures T_1, T_2, and T_3.
    - **Statement of the law of additive voltages**: For a given set of 3 thermocouple wires, A, B, and C, all measuring the same temperature difference $T_1 - T_2$, the voltage measured by wires A and C must equal the sum of the voltage measured by wires A and B and the voltage measured by wires B and C, i.e., $V_{1-2,\text{A\&C}} = V_{1-2,\text{A\&B}} + V_{1-2,\text{B\&C}}$.
- Thermopile – Several thermocouples in series.
- Thermoresistive devices – RTDs (metal, R increases with increasing T), thermistors (semiconductor, R decreases with increasing T).
- Radiative devices – e.g., infrared pyrometer: $E = \varepsilon\sigma T^4$, $\sigma = 5.669 \times 10^{-8}\ \frac{\text{W}}{\text{m}^2\text{K}^4}$, $T_H = \left(\frac{\varepsilon_{\text{assumed}}}{\varepsilon_{\text{actual}}}\right)^{1/4} T_{\text{ind}}$.
- Tables – how to use thermocouple tables and RTD tables, and interpolation. Reference temperature is 0°C for all temperature tables.

Measurement of Mechanical Quantities

- Position – mechanical devices; interferometer; potentiometer; linear variable displacement transducer (LVDT); ultrasonic transducer: pulse-echo $x = \frac{a\Delta t}{2}$, through transmission $x = a\Delta t$; capacitance sensor: $C = K\varepsilon_0 A/d$, $\varepsilon_0 = 8.854\times10^{-12}\ \frac{\mathrm{C}^2}{\mathrm{N\cdot m}^2}$; laser displacement meter.
- Angular velocity and rpm – $\left(N_{\mathrm{rpm}}\frac{\text{rotation}}{\text{min}}\right) = \left(\omega\frac{\text{radian}}{\text{s}}\right)\left(\frac{\text{rotation}}{2\pi\ \text{radian}}\right)\left(\frac{60\ \text{s}}{\text{min}}\right)$; contacting tachometer, noncontacting tachometer: $N_{\mathrm{rpm}} = \frac{P}{n_{\mathrm{teeth}}}$. For stroboscopic tachometers, watch out for *aliasing*.
- Torque and power – $\dot{W}_{\mathrm{shaft}} = \omega T = \frac{2\pi}{60}N_{\mathrm{rpm}}T$; dynamometers: prony brake, cradled DC motor, eddy current.

Pressure Measurement

- Types of pressure – absolute, gage: $P_{\mathrm{gage}} = P_{\mathrm{abs}} - P_{\mathrm{atm}}$, vacuum: $P_{\mathrm{vac}} = P_{\mathrm{atm}} - P_{\mathrm{abs}}$, $P_{\mathrm{vac}} = -P_{\mathrm{gage}}$.
- Hydrostatics – $P_{\mathrm{below}} = P_{\mathrm{above}} + \rho g|\Delta z|$; applications include:
  - Liquid manometers (U-tube manometers): typically, $\Delta P = (\rho_{\mathrm{m}} - \rho)gh$ if measurement locations are at the same elevation on either side of the manometer.
  - McCleod gage – a special type of liquid manometer used to measure very low pressures.
  - Hydraulic jacks – the ideal mechanical advantage is $\frac{F_2}{F_1} = \frac{A_2}{A_1}$.
- Mechanical pressure gages – work by mechanical means, without any electricity or liquid columns:
  - Bourdon tube – flattened hollow tube that is coiled, bent, or twisted.
  - Deadweight tester – weights applied to a piston of known area, thus $P = F/A_e$.
- Electronic pressure transducers – diaphragm type, with displacement measured by:
  - Strain gage – a strain gage is mounted on the diaphragm itself, sensing strain in the diaphragm.
  - Capacitance – the diaphragm is mounted close to a fixed parallel plate, capacitance is measured.
  - LVDT – diaphragm is attached to the core of a linear variable displacement transducer.
  - Optical – various optical techniques used to measure the degree of diaphragm deformation.
- Piezoelectric pressure transducers – compression causes a voltage that can be calibrated with pressure.

Velocity Measurement

- Linear velocity transducer (LVT) – Similar in principle to LVDT, but measure velocity, not displacement.
- Doppler radar velocimeter – Doppler frequency shift $\Delta f_D = \frac{2V\cos\theta}{\lambda}$.
- Displacement sensors – $V(t) = dx(t)/dt$, *noise is amplified by differentiation*.
- Acceleration sensors – $V(t) = V_0 + \int_{t_0}^{t} a(t)dt$, *noise is attenuated by integration*.
- Velocity of fluids – Lagrangian methods follow fluid particle moving with the flow: $V(t) = dx(t)/dt$; Eulerian methods measure velocity field with a probe sitting in the flow.
- Laser Doppler velocimeter (LDV) – particles in fringe pattern of focal volume, $V = fs = \frac{f\lambda}{2\sin(\alpha/2)}$.
- Particle image velocimetry (PIV) – velocity vectors calculated from moving particles in the plane illuminated by a laser sheet, $V = \Delta s/\Delta t$.
- Hot-wire and hot-film anemometer – wire heated to constant temperature; King's law $E^2 = a + bV^n$.
- Pitot-static probe – Pitot formula $V = \sqrt{\frac{2(P_1 - P_2)}{\rho}}$.
- Other – rotating velocimeter (cup anemometer, turbine or vane anemometer), electromagnetic velocimeter.

Volume Flow Rate Measurement

- Mass flow rate vs. volume flow rate – $\dot{m} = \rho \dot{V}$. *Notation*: V = volume, V = velocity, $\dot{V}$ = volume flow rate.
- End-line measurement – $\dot{V} = V / \Delta t$ vs. in-line measurement – various types:
- Obstruction – orifice, flow nozzle, and Venturi: $\beta = d / D$, $\dot{V} = A_o C_d \sqrt{\frac{2(P_1 - P_2)}{\rho(1-\beta^4)}}$, where d = orifice diameter, D = pipe diameter, A_o = orifice area, V_1 = average speed through the pipe, V_2 = average speed through the orifice (or throat), $P_1 - P_2$ = measured pressure drop from upstream to downstream, C_d = discharge coefficient, and Reynolds number is based on the *pipe* not the orifice, i.e., $\text{Re} = \frac{\rho V_1 D}{\mu} = \frac{V_1 D}{\nu}$:
    - For a standard orifice flowmeter, $C_d = 0.5959 + 0.0312\beta^{2.1} - 0.184\beta^{8.0} + 91.71\beta^{2.5} / \text{Re}^{0.75}$
    - For a standard flow nozzle flowmeter, $C_d = 0.9975 - 6.53\beta^{0.50} / \text{Re}^{0.50}$
    - For a standard Venturi flowmeter, $C_d = 0.98$
- Laminar flow element – pressure drop calibrated against volume flow rate, typically linear calibration because flow through the small tubes is laminar even though flow through the pipe may be turbulent.
- Positive displacement – measure how many "trapped" or "captured" volumes pass through per unit time.
- Turbine and paddlewheel – spin a turbine or paddlewheel connected to a shaft, measure rpm and calibrate.
- Roatmeters or floatmeters or variable-area flowmeters – float hovers when forces balance.
- Miscellaneous flowmeters – ultrasonic (transit time and Doppler shift), electromagnetic, vortex, others.