How to iterate to find \( V_t \) (gravitational terminal settling speed)

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Example (this example is for gravimetric settling; a similar procedure is used for inertial separation)

Given:
- Spherical particles of 100 \( \mu \text{m} \) diameter with particle density = 1600 kg/m\(^3\).
- Gravimetric settling with \( g = 9.807 \text{ m/s}^2 \).
- Air at SATP: \( \rho = 1.184 \text{ kg/m}^3 \) and \( \mu = 1.849 \times 10^{-5} \text{ kg/(m s)} \).

To do: Calculate the terminal settling speed for these particles as they fall in quiescent air due to gravity.

Solution: The equations we need are:

\[
\text{Kn} = \frac{\lambda}{D_p}, \quad \lambda = \frac{\mu}{0.499 \sqrt{8 \rho P}}, \quad C = 1 + \text{Kn} \left[ 2.514 + 0.80 \exp \left( -\frac{0.55}{\text{Kn}} \right) \right], \quad C_D = C_D(\text{Re}), \quad \text{where } \text{Re} = \frac{\rho V_t D_p}{\mu}.
\]

There are many equations for the drag coefficient of a sphere:

- \( C_D = \frac{24}{\text{Re}} \) for \( \text{Re} < 0.1 \) (Stoke’s flow),
- \( C_D = \frac{24}{\text{Re}} \left( 1 + 0.0916 \text{Re} \right) \) for any \( \text{Re} < 5 \),
- \( C_D = \frac{24}{\text{Re}} \left( 1 + 0.158 \text{Re}^{0.5} \right) \) for \( 5 < \text{Re} < 1000 \),
- \( C_D = 0.4 + \frac{24}{\text{Re}} \frac{6}{1 + \sqrt{\text{Re}}} \) for \( 1000 < \text{Re} < 10^5 \),
- \( C_D \approx 0.2 \) for \( \text{Re} > 2 \times 10^6 \).

Note that \( C_D \) is a strong function of surface roughness for \( \text{Re} \) between about \( 10^5 \) and \( 10^6 \).

Also, the equations for terminal settling speed for gravimetric settling are

\[
V_t = \sqrt[3]{\frac{4}{3} \frac{\rho_p - \rho}{\rho g D_p} \frac{C}{C_D}}, \quad \text{Re} = \frac{\rho V_t D_p}{\mu}.
\]

In our example, \( \lambda = \frac{\mu}{0.499 \sqrt{8 \rho P}} = \frac{1.849 \times 10^{-5} \text{ kg/(m s)}}{0.499 \sqrt{8 (1.184 \text{ kg/m}^3)(101325 \text{ N/m}^3)}} = 0.06704 \mu\text{m}, \)

\[
\text{Kn} = \frac{\lambda}{D_p} = \frac{0.06704 \mu\text{m}}{100 \mu\text{m}} = 0.0006704, \quad C = 1 + 0.0006704 \left[ 2.514 + 0.80 \exp \left( -\frac{0.55}{0.0006704} \right) \right] = 1.00168.
\]

Now we need to set up our iteration. First we guess \( V_t \). Stoke’s approximation is typically a good first guess, but you can pick any guess you want – it just might take longer to converge. Stokes’ approximation is

\[
V_t = \frac{\rho_p - \rho}{18} D_p^2 g \frac{C}{\mu} = \frac{1600 - 1.184 \text{ kg/m}^3}{18} \left( 100 \times 10^{-6} \text{ m} \right)^2 \left( 9.807 \text{ m/s}^2 \right) \frac{1.00168}{1.849 \times 10^{-5} \text{ kg/(m s)}} = 0.47191 \text{ m/s}.
\]

This Stokes value is used as a first guess. We set up a table to iterate, using the following equations in sequence, and then using the new \( V_t \) as our next guess:

\[
\text{Re} = \frac{\rho V_t D_p}{\mu}, \quad C_D = \frac{24}{\text{Re}} \left( 1 + 0.0916 \text{Re} \right), \quad V_t = \sqrt[3]{\frac{4}{3} \frac{\rho_p - \rho}{\rho g D_p} \frac{C}{C_D}}.
\]

Note: We use the drag coefficient equation for \( \text{Re} < 5 \). If \( \text{Re} > 5 \), a different equation would need to be used.

This kind of repetitive calculation or iteration is most easily done in Excel, but I show it here “by hand” in a table.
Notice that we use the new value of velocity as the next guess:
<table>
<thead>
<tr>
<th>$V_t$ (m/s)</th>
<th>Re</th>
<th>$C_D$</th>
<th>New $V_t$ (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4719</td>
<td>3.022</td>
<td>10.14</td>
<td>0.4176</td>
</tr>
<tr>
<td>0.4176</td>
<td>2.674</td>
<td>11.173</td>
<td>0.3979</td>
</tr>
<tr>
<td>0.3979</td>
<td>2.548</td>
<td>11.618</td>
<td>0.39018</td>
</tr>
<tr>
<td>0.39018</td>
<td>2.4985</td>
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<tr>
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<td>11.925</td>
<td>0.38512</td>
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<tr>
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<td>2.4661</td>
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<tr>
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</tr>
<tr>
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<td>11.933</td>
<td>0.38499</td>
</tr>
<tr>
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<td>2.4653</td>
<td>11.934</td>
<td>0.38498</td>
</tr>
<tr>
<td>0.38498</td>
<td>2.4652</td>
<td>11.934</td>
<td>0.38498</td>
</tr>
</tbody>
</table>

We see that we have converged to 5 significant digits in $V_t$. We note that 3 significant digits is probably about the best we can hope for in this kind of exercise. Nevertheless, we write the final values to 4 significant digits below:

Re = 2.465, $C_D$ = 11.93, $V_t$ = 0.3850 m/s