Today, we will:

- Finish talking about gradient diffusion and discuss the Reynolds analogy

Examples of the One-Dimensional Diffusion Equation:

<table>
<thead>
<tr>
<th>$A$ and $a$ (the property with a gradient)</th>
<th>$J_A$ (the amount of $A$ diffused per unit area per unit time)</th>
<th>$b$ (diffusion coefficient)</th>
<th>1-D gradient diffusion equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A = \text{energy} = mC_pT$</td>
<td>$J_A = q = \text{heat flux}$</td>
<td>$\kappa = \text{thermal diffusivity}$</td>
<td>$J_A = -b\frac{da}{dz}$</td>
</tr>
<tr>
<td>$a = \frac{A}{V} = \frac{m}{V}C_pT = \rho C_pT$</td>
<td>$q = \text{rate of heat (energy) transfer per unit area}$</td>
<td>$\kappa = \frac{k}{\rho C_p}$</td>
<td>$q = -\kappa \frac{d}{dz}(\rho C_pT)$</td>
</tr>
<tr>
<td>$a = \rho C_pT$</td>
<td>${q} = \left{\frac{\text{energy}}{\text{area} \cdot \text{time}}\right}$</td>
<td>${\kappa} = \left{\frac{\text{length}^2}{\text{time}}\right}$</td>
<td>or $q = -k \frac{dT}{dz}$</td>
</tr>
<tr>
<td>So, we have a gradient of temperature</td>
<td></td>
<td></td>
<td>Heat diffusion equation</td>
</tr>
</tbody>
</table>

| $A = \text{magnitude of momentum} = mU$ | $J_A = -\tau = \text{shear stress}$                           | $\nu = \text{kinematic viscosity}$ | $-\tau = -\nu \frac{d}{dz}(\rho U)$ |
| $a = \frac{A}{V} = \frac{m}{V}U = \rho U$ | $-\tau = \frac{d\{\text{momentum}\}}{dt} \frac{1}{\text{area}}$ | $\nu = \frac{\mu}{\rho}$ | or $-\tau = -\mu \frac{dU}{dz}$ |
| $a = \rho U$                            | $\{-\tau\} = \left\{\frac{\text{momentum}}{\text{area} \cdot \text{time}}\right\}$ | $\{\nu\} = \left\{\frac{\text{length}^2}{\text{time}}\right\}$ | Equation for 1-D shear stress |
| So, we have a gradient of magnitude of velocity or speed |

| $A = \text{number of mols of species } j = n_j$ | $J_A = J_j = \text{molar flux}$                             | $D_{aj} = \text{binary diffusion coefficient between air and species } j$ | $J_j = -D_{aj} \frac{d}{dz}(c_{\text{molar},j})$ |
| $a = \frac{n_j}{V} = c_{\text{molar},j}$ | $J_j = \text{rate of transfer of mols of species } j \text{ per unit area}$ | $\{D_{aj}\} = \left\{\frac{\text{length}^2}{\text{time}}\right\}$ | Fick’s law in terms of molar concentration |
| $a = c_{\text{molar},j}$                      | $\{J_j\} = \left\{\frac{\text{mols}}{\text{area} \cdot \text{time}}\right\}$ | |
| So, we have a gradient of species (mols, but we can also think of it as a gradient of species mass) |

Fick’s law in terms of mass concentration $c_j$ (multiply both sides by $M_j$):

$$M_j J_j = -D_{aj} \frac{dc_j}{dz}$$
**Nondimensional Ratios of Diffusion Coefficients:**

Recall, when the one-dimensional diffusion equations for energy, momentum, and mass (species) are written properly, the diffusion coefficient for each case has the same dimensions. Namely:

**Energy (heat):**

\[ q = -\kappa \frac{d}{dz} \left( \rho C_p T \right) \]

\[ \{\kappa\} = \left\{ \frac{\text{length}^2}{\text{time}} \right\} \]

**Momentum:**

\[ -\tau = -\nu \frac{d}{dz} (\rho U) \]

\[ \{\nu\} = \left\{ \frac{\text{length}^2}{\text{time}} \right\} \]

**Species:**

\[ J_j = -D_{aj} \frac{d}{dz} \left( c_{\text{molar},j} \right) \]

\[ \{D_{aj}\} = \left\{ \frac{\text{length}^2}{\text{time}} \right\} \]

So… since all three of these coefficients have the same dimensions, we can create nondimensional parameters out of them by taking ratios of any two of them:

We define the following nondimensional *laminar* diffusion ratios:

- **Sc** = Schmidt number:  \[ Sc = \frac{\nu}{D_{aj}} \] (ratio of momentum diffusion to species diffusion)

- **Le** = Lewis number:  \[ Le = \frac{\kappa}{D_{aj}} \] (ratio of heat energy diffusion to species diffusion)

- **Pr** = Prandtl number:  \[ Pr = \frac{\nu}{\kappa} \] (ratio of momentum diffusion to heat energy diffusion)

How to interpret these ratios? Take Schmidt number as an example:

- If Sc is *small*, momentum diffuses much more *slowly* than species (\( \nu \ll D_{aj} \)).
- If Sc is *large*, momentum diffuses much more *rapidly* than species (\( \nu \gg D_{aj} \)).

**Example: Physical significance of Prandtl number**

**Given:** A laminar boundary layer on a flat plate with velocity and thermal BLs sketched.

**To do:** Choose the correct description of Prandtl number for this fluid:

- A. \( Pr < 1 \)
- B. \( Pr \approx 1 \)
- C. \( Pr > 1 \)
Reynolds analogy for laminar cases – Energy, momentum, and mass, all diffuse in similar fashion, due to molecular diffusion. Compare:

**Suddenly heated wall** $[T = T_0 = 0^\circ \text{C} \text{ everywhere, then suddenly } T = T_i \text{ at the wall.}]$

- **Energy** is diffused upward.
- Rate of diffusion depends on $\kappa$ (thermal diffusivity).

**Suddenly moving wall** $[U = U_0 = 0 \text{ m/s} \text{ everywhere, then suddenly } U = U_i \text{ at the wall.}]$

- **Momentum** is diffused upward.
- Rate of diffusion depends on $\nu$ (kinematic viscosity).

**Sudden removal of a membrane** $[c_{\text{molar}} = c_{\text{molar},0} = 0 \text{ mol/m}^3 \text{ everywhere, then suddenly } c_{\text{molar}} = c_{\text{molar},i} \text{ at the location of the membrane, and the membrane disappears suddenly}].]$

- **Mass** (species) is diffused upward.
- Rate of diffusion depends on $D_{aj}$ (binary diffusion coefficient).
Reynolds analogy for turbulent cases – Energy, momentum, and mass, all diffuse in similar fashion, due to large turbulent eddies which promote rapid mixing.

For the turbulent case, we have the same analogy between energy, momentum, and mass (species) that we discussed for the laminar case, except: Instead of gas molecules randomly moving around on a microscopic scale (molecular diffusion), we have large turbulent eddies moving around on a much larger scale than that of molecular diffusion (macroscopic diffusion). Mixing is determined by the turbulent eddies.

Laminar diffusion (molecular)  
Turbulent diffusion (large eddies)

So, for the turbulent case, we expect the diffusion coefficients to be much larger than those of the laminar case:

- We expect turbulent kinematic viscosity = $\nu_t$ $>>$ $\nu$
- We expect turbulent thermal diffusivity = $\kappa_t$ $>>$ $\kappa$
- We expect turbulent binary diffusion coefficient = $D_{aj,t}$ $>>$ $D_{aj}$

Usefulness of Reynolds analogy:

- We can easily measure heat transfer characteristics, but mass transfer is harder to measure.
- So, we use the heat transfer correlations (Nusselt number vs. Reynolds number, etc.) to predict mass transfer behavior – they are analogous by the Reynolds analogy.
- This is what is often done in air pollution work.

Let’s define the following nondimensional turbulent diffusion ratios:

- $Sc_t$ = turbulent Schmidt number: $Sc_t = \frac{V_t}{D_{aj,t}}$
- $Le_t$ = turbulent Lewis number: $Le_t = \frac{K_t}{D_{aj,t}}$
- $Pr_t$ = turbulent Prandtl number: $Pr_t = \frac{V_t}{K_t}$
Review question:
Given: We know that we emit carbon dioxide into the atmosphere when we drive our cars. Approximate gasoline as mostly octane, $C_8H_{18}$. The molecular weight of gasoline is about 110 kg/kmol and its specific gravity is about 0.75. The molecular weight of CO$_2$ is about 44.0 kg/kmol.

To do: Approximating the combustion of gasoline as ideal combustion of octane, estimate how many kg of CO$_2$ you emit into the atmosphere for every gallon of gasoline you burn.

Solution: This conversion factor is useful: \(1 \text{ m}^3 = 264.17 \text{ gallons}\).