Today, we will:

- Quickly review Lecture 13 since Professor Cimbala had oral surgery on Wednesday
- Begin to discuss the **Gaussian plume model**: how pollutants diffuse in plumes
**Reynolds analogy for turbulent cases** – Energy, momentum, and mass, all diffuse in similar fashion, due to large turbulent eddies which promote rapid mixing.

For the **turbulent** case, we have the same analogy between energy, momentum, and mass (species) that we discussed for the laminar case, except:

Instead of gas molecules randomly moving around on a *microscopic* scale (*molecular diffusion*), we have large turbulent eddies moving around on a much larger scale than that of molecular diffusion (*macroscopic diffusion*). *Mixing is determined by the turbulent eddies.*

![Diagram: Laminar diffusion (molecular) vs. Turbulent diffusion (large eddies)](image)

So, for the **turbulent** case, we expect the diffusion coefficients to be **much larger** than those of the laminar case:

- We expect **turbulent** kinematic viscosity = \( \nu_t \gg \nu \)
- We expect **turbulent** thermal diffusivity = \( \kappa_t \gg \kappa \)
- We expect **turbulent** binary diffusion coefficient = \( D_{aj,t} \gg D_{aj} \)

Usefulness of Reynolds analogy for turbulent diffusion:

- We can more easily measure heat transfer behavior than mass transfer behavior.
- So, we use the *heat* transfer correlations (Nusselt number vs. Reynolds number, etc.) to predict *mass* transfer behavior – they are analogous by the Reynolds analogy.
- This is what is often done in air pollution work.

Let’s define the following **nondimensional turbulent diffusion ratios** (subscript \( t \) added):

- **\( Sc_t \) = turbulent Schmidt number:** \( Sc_t = \frac{V_t}{D_{aj,t}} \)
- **\( Le_t \) = turbulent Lewis number:** \( Le_t = \frac{\kappa_t}{D_{aj,t}} \)
- **\( Pr_t \) = turbulent Prandtl number:** \( Pr_t = \frac{V_t}{\kappa_t} \)
The **Gaussian Plume Model** – useful for predicting the air pollution concentration downwind of a smoke stack.

First (simplest) approximation: A point source of contaminant, with no buoyancy effects:

What do you think of this old saying: “The **solution to pollution is dilution**!”
3-D sketch of a buoyant plume

\[ H = h_z + \delta h \]
Let’s do a **mass balance** of species \( j \) on a small element of air in the plume of dimensions \( dx, dy, \) and \( dz \) as sketched here.

- We use **truncated Taylor series expansions** (truncated to first order, neglecting higher-order terms).
- Consider the \( x \) direction first (the same direction as the wind).
- There are two parts of the mass flow rate coming in or out: **advection** (due to the air flow in and out) and **diffusion** (due to species diffusion as we have been discussing).

\[
\begin{align*}
\dot{m}_{j,in,x} &= U c_j \, dydz \\
- D_{aj,x} \frac{\partial c_j}{\partial x} \, dydz \\
\end{align*}
\]

\[
\dot{m}_{j,out,x} = U c_j \, dydz + \frac{\partial (U c_j)}{\partial x} \, dx dydz \\
- D_{aj,x} \frac{\partial c_j}{\partial x} \, dydz + \frac{\partial (D_{aj,x} \frac{\partial c_j}{\partial x})}{\partial x} \, dx dydz
\]

- Now consider the \( y \) direction in which there is **no wind**.
- Thus, there is **diffusion** in the \( y \)-direction, but **no advection**.
- **Note**: The diffusion coefficient \( D_{aj,y} \) in the \( y \) direction may be **different** than the diffusion coefficient \( D_{aj,x} \) in the \( x \) direction.

\[
\begin{align*}
\dot{m}_{j,out,y} &= - \left[ D_{aj,y} \frac{\partial c_j}{\partial y} \, dxzdz + \frac{\partial (D_{aj,y} \frac{\partial c_j}{\partial y})}{\partial y} \, dydxdz \right] \\
\end{align*}
\]

\[
\dot{m}_{j,in,y} = - D_{aj,y} \frac{\partial c_j}{\partial y} \, dxdz
\]
Finally, consider the $z$ direction in which there also is **no wind**.

Thus, there is **diffusion** in the $z$-direction, but **no advection**.

**Note:** The diffusion coefficient $D_{aj,z}$ in the $z$ direction may be different than the diffusion coefficient $D_{aj,x}$ in the $x$ direction or $D_{aj,y}$ in the $y$ direction.

$$
\dot{m}_{j,\text{out,z}} = -
\begin{bmatrix}
D_{aj,z} \frac{\partial c_j}{\partial z} dxdy \\
\partial \left( D_{aj,z} \frac{\partial c_j}{\partial z} \right) dzdydx
\end{bmatrix}
$$

Now consider **conservation of mass of species $j$** in our little control volume. We write

$$
\frac{dm_j}{dt} = \sum_{\text{in}} \dot{m}_j - \sum_{\text{out}} \dot{m}_j
$$

The above equation says that the rate of change of mass of species $j$ within the little control volume must equal the net mass flow rate of species $j$ into the control volume.

But since mass of species $j$ is equal to mass concentration times volume, the LHS becomes

$$
\frac{dm_j}{dt} = \nabla \cdot d\varepsilon_j = dxdydz \frac{dc_j}{dt}
$$

To get the RHS, we add up all the mass flow rates of species $j$ (three flows in and three flows out) to get the net rate of mass flow of species $j$ into the little control volume,

$$
\sum_{\text{in}} \dot{m}_j - \sum_{\text{out}} \dot{m}_j = (\dot{m}_{j,\text{in,x}} + \dot{m}_{j,\text{in,y}} + \dot{m}_{j,\text{in,z}}) - (\dot{m}_{j,\text{out,x}} + \dot{m}_{j,\text{out,y}} + \dot{m}_{j,\text{out,z}})
$$

Putting everything together,
\[
dx dy dz \frac{dc_j}{dt} = (m_{j,in,x} + m_{j,in,y} + m_{j,in,z}) - (m_{j,out,x} + m_{j,out,y} + m_{j,out,z})
\]
\[
= \left( Uc_j dy dz - D_{aj,x} \frac{\partial c_j}{\partial x} dy dz - D_{aj,y} \frac{\partial c_j}{\partial y} dx dz - D_{aj,z} \frac{\partial c_j}{\partial z} dx dy \right)
\]
\[
- \left[ \begin{array}{c}
Uc_j dy dz + \frac{\partial (Uc_j)}{\partial x} dx dy dz - D_{aj,x} \frac{\partial c_j}{\partial x} dy dz + \frac{\partial \left( D_{aj,x} \frac{\partial c_j}{\partial x} \right)}{\partial x} dx dy dz \\
- D_{aj,y} \frac{\partial c_j}{\partial y} dx dz + \frac{\partial \left( D_{aj,y} \frac{\partial c_j}{\partial y} \right)}{\partial y} dy dz \\
- D_{aj,z} \frac{\partial c_j}{\partial z} dz dx + \frac{\partial \left( D_{aj,z} \frac{\partial c_j}{\partial z} \right)}{\partial z} dz dy dx
\end{array} \right]
\]

- Some terms cancel, and all the remaining terms contain \(dx dy dz\), which can be removed from both sides of the equation, leaving us with

\[
\frac{\partial c_j}{\partial t} = -\frac{\partial}{\partial x} (Uc_j) + \frac{\partial}{\partial x} \left( D_{aj,x} \frac{\partial c_j}{\partial x} \right) + \frac{\partial}{\partial y} \left( D_{aj,y} \frac{\partial c_j}{\partial y} \right) + \frac{\partial}{\partial z} \left( D_{aj,z} \frac{\partial c_j}{\partial z} \right)
\]