# M E 433 Professor John M. Cimbala Lecture 24

**Today, we will**:

- Summarize our equations for particle motion
- Discuss **terminal settling velocity** and equations for its prediction in quiescent air
- Discuss **aerodynamic equivalent diameter** and how to apply it

**Review of equations for particle motion so far:**

**Relative particle velocity** = $\boxed{\vec{v}_r = \vec{v} - \vec{U}}$, where $\vec{v}$ = particle velocity and $\vec{U}$ = air velocity.

**Drag force on a spherical particle** = $\boxed{\vec{F}_{\text{drag}} = -\frac{1}{8}\rho\frac{C_D}{C}\pi D_p^{\,2}\vec{v}_r\left|\vec{v}_r\right|}$, where

Cunningham correction factor

Knudsen

$\boxed{C = 1 + \text{Kn}\left[2.514 + 0.80\exp\left(-\frac{0.55}{\text{Kn}}\right)\right]}$, $\boxed{\text{Kn} = \frac{\lambda}{D_p}}$, $\boxed{\lambda = \frac{\mu}{0.499}\sqrt{\frac{\pi}{8\rho P}}}$, $\boxed{\text{Re} = \frac{\rho\left|\vec{v}_r\right|D_p}{\mu}}$, and

$\boxed{C_D = \frac{24}{\text{Re}} \text{ for Re} < 0.1}$, $\boxed{C_D = \frac{24}{\text{Re}}\left(1 + 0.0916\,\text{Re}\right) \text{ for } 0.1 < \text{Re} < 5}$.

Drag coeff

$\Sigma\vec{F}$ on particle

**Equation of motion for spherical particle**: $\boxed{m_p\frac{d\vec{v}}{dt} = \frac{\pi D_p^{\,3}}{6}\left(\rho_p - \rho\right)\vec{g} - \frac{1}{8}\rho\frac{C_D}{C}\pi D_p^{\,2}\vec{v}_r\left|\vec{v}_r\right|}$.

Newton's 2nd law → $\vec{a}_p$

But for a spherical particle, we also know its mass, $\boxed{m_p = \rho_p V_p = \rho_p\frac{\pi D_p^{\,3}}{6}}$. Plugging this mass into the equation of motion (after a little algebra), we get our final expression,

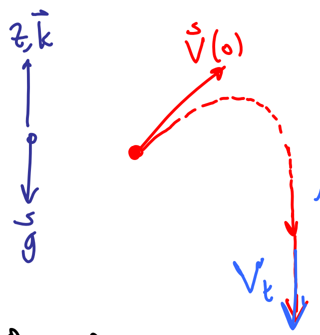
$$\boxed{\frac{d\vec{v}}{dt} = \frac{\rho_p - \rho}{\rho_p}\vec{g} - \frac{3}{4}\frac{\rho}{\rho_p}\frac{C_D}{C}\frac{1}{D_p}\vec{v}_r\left|\vec{v}_r\right|}.$$

= 0 when $\vec{v}$ = constant

no air movement

**Simplest application – <u>Terminal settling velocity in quiescent air</u>** → $\vec{U} = 0$

Let $\vec{U} = 0$ → let $V_t$ = terminal settling velocity




eventually $\boxed{\vec{V} = -V_t\vec{k}}$ → falls down at a steady velocity

Gravity force exactly balances drag force after some time

Particle then falls at speed $V_t$ = constant

no acceleration

Our Diff eq becomes $\boxed{0 = (\rho_p - \rho)g - \frac{3}{4}\rho\frac{C_D}{C}\frac{1}{D_p}V_t^2}$ in vertical direction (LHS = 0)

Solve for $V_t$

**Terminal settling velocity**: $\boxed{V_t = \sqrt{\frac{4}{3}\frac{\rho_p - \rho}{\rho}gD_p\frac{C}{C_D}}}$, but $C_D = C_D(\text{Re})$, where $\boxed{\text{Re} = \frac{\rho V_t D_p}{\mu}}$.

implicit Eq.

Calculations @ STP: air $\rho = 1.184\ kg/m^3$

$\mu = 1.849 \times 10^{-5}\ kg/m \cdot s$

Easiest case is Stokes flow $\longrightarrow$ If $Re \lesssim 0.1$, $\boxed{C_D = \frac{24}{Re} = \frac{24\mu}{\rho V_t D_p}}$

Plug into above eq.

Square it $\rightarrow$ $V_t^2 = \frac{4}{3}\left(\frac{\rho_p - \rho}{\cancel{\rho}}\right) g D_p \frac{C \cancel{\rho} V_t D_p}{24\mu}$

Solve for $V_t$ $\longrightarrow$ $\boxed{V_t = \frac{\rho_p - \rho}{18} D_p^2 g \frac{C}{\mu}}$

★ Terminal settling velocity for Stokes flow ($Re \lesssim 0.1$)

for spheres.

But it does a reasonable job up to $Re \sim 1$

**Example: Terminal settling velocity as a function of particle diameter**

**Given**: Air at STP with Cunningham correction factors from previous calculations. For Stokes flow (Re less than about 0.1), $\boxed{C_D = 24/\text{Re}}$, and $\boxed{V_t = \frac{\rho_p - \rho}{18}\left(D_p^{\ 2}\right) g \frac{C}{\mu}}$.

**To do**:
Calculate $V_t$ for various values of particle diameter $D_p$. Also calculate Re to test our Stokes flow approximation. At STP conditions, $\rho$ = 1.184 kg/m$^3$, and $\mu$ = 1.849 × 10$^{-5}$ kg/(m s). Use $g$ = 9.807 m/s$^2$ (gravitational constant), and $\rho_p$ = 1000 kg/m$^3$ (unit density spheres). [Give your answers to 3 significant digits, and be careful with units.]

**Solution**:

Enter " $D_p$, $V_t$, Re "

**Table to be filled in during class:**

| $D_p$ (μm) | $C$ | $V_t$ (m/s) | Re |
|---|---|---|---|
| 0.001 | 222.7 | | |
| 0.0025 | 89.43 | $1.64 \times 10^{-8}$ | $2.63 \times 10^{-12}$ |
| 0.006 | 37.60 | $3.98 \times 10^{-8}$ | $1.53 \times 10^{-11}$ |
| 0.01 | 22.79 | $6.71 \times 10^{-8}$ | $4.29 \times 10^{-11}$ |
| 0.025 | 9.489 | | |
| 0.06 | 4.355 | | |
| 0.1 | 2.921 | | |
| 0.25 | 1.702 | | |

| $D_p$ (μm) | $C$ | $V_t$ (m/s) | Re |
|---|---|---|---|
| 0.6 | 1.282 | $1.36 \times 10^{-5}$ | |
| 1 | 1.169 | | |
| 2.5 | 1.067 | | |
| 6 | 1.028 | | |
| 10 | 1.017 | 0.00299 | 0.00192 |
| 25 | 1.007 | 0.0185 | 0.0296 |
| 60 | 1.003 | 0.106 | 0.408 |
| 100 | 1.0017 | 0.295 | 1.89 |

Sample calc. @ $D_p$ = 100 μm

$$V_t = \frac{\rho_p - \rho}{18} D_p^2 \, g \frac{C}{\mu} = \frac{(1000 - 1.184)\ \text{kg/m}^3}{18}\left(100\times10^{-6}\ \text{m}\right)^2 \left(9.807\ \frac{\text{m}}{\text{s}^2}\right)\left(\frac{1.0017}{1.849\times10^{-5}}\ \frac{\text{m}\cdot\text{s}}{\text{kg}}\right)$$

$$V_t = 0.295\ \text{m/s}$$

$$Re = \frac{\rho V_t D_p}{\mu} = \frac{(1.184\ \text{kg/m}^3)(0.29482\ \text{m/s})(100\times10^{-6}\ \text{m})}{1.849\times10^{-5}\ \text{kg/m}\cdot\text{s}}$$

$$Re = 1.8878$$

> 0.1 – so Stokes is not valid

Need to do some iteration to get a better answer

**Filled in Table:**

| $D_p$ (μm) | $C$ | $V_t$ (m/s) | Re | $D_p$ (μm) | $C$ | $V_t$ (m/s) | Re |
|---|---|---|---|---|---|---|---|
| 0.001 | 222.7 | 6.56E-09 | 4.20E-13 | 0.6 | 1.282 | 1.36E-05 | 5.22E-07 |
| 0.0025 | 89.43 | 1.65E-08 | 2.63E-12 | 1 | 1.169 | 3.44E-05 | 2.20E-06 |
| 0.006 | 37.60 | 3.98E-08 | 1.53E-11 | 2.5 | 1.067 | 1.96E-04 | 3.14E-05 |
| 0.01 | 22.79 | 6.71E-08 | 4.30E-11 | 6 | 1.028 | 1.09E-03 | 4.19E-04 |
| 0.025 | 9.489 | 1.75E-07 | 2.79E-10 | 10 | 1.017 | 2.99E-03 | 1.92E-03 |
| 0.06 | 4.355 | 4.61E-07 | 1.77E-09 | 25 | 1.007 | 1.85E-02 | 2.96E-02 |
| 0.1 | 2.921 | 8.60E-07 | 5.51E-09 | 60 | 1.003 | 1.06E-01 | 4.08E-01 |
| 0.25 | 1.702 | 3.13E-06 | 5.01E-08 | 100 | 1.0017 | 0.295 | 1.89 |

Notice that $V_t \uparrow$ rapidly as $D_p \uparrow$

When Re $\gtrsim$ 0.1 we need to iterate using the other $C_D$ eq.

For $0.1 \lesssim Re \lesssim 5$, $\boxed{C_D = \frac{24}{Re}\left(1 + 0.0916\, Re\right)}$

Use

$$\boxed{V_t = \sqrt{\frac{4}{3}\frac{(\rho_p - \rho)}{\rho} g D_p \frac{C}{C_D}}}$$

@ $D_p$ = 100 μm

Iteration Procedure

| Re | $C_D$ | $V_t$ (m/s) |
|---|---|---|
| 1.8878 | 14.912 | 0.27221 |
| 1.743 | | |
| ⋮ | | |
| 1.641 | 16.82 | 0.2563 |

First guess comes from Stokes solution (see Table) → 1.8878

0.27221 → calc. a new Re from this

iterate for several iterations

Converged values

$V_t = 0.2948$ m/s

@ 100 μm

(≈ 15% error compared to Stokes)

For $D_p \lesssim 10\,\mu m$ (typ. air pollution), Stokes does a pretty good job.

Applications:

- **Aerodynamic Equivalent Diameter** → non spherical particles
  – may be non-unit density ($\rho \neq 1000\ kg/m^3$)

$D_{ae}$ = aero. equiv. diameter ≡ diameter of a spherical unit-density particle that has the same terminal settling velocity as the actual particle

To calculate $D_{ae}$

($\rho_0$ = Unit density $= 1000\ kg/m^3$)

- measure $V_t$ for the actual particle
- Use eq for $V_t$ → $V_t = \sqrt{\frac{4}{3}\frac{(\rho_p - \rho)}{\rho} g D_p \frac{C}{C_D}}$

$\rho_0$

$= D_{ae}$

with $\rho_p = \rho_0$ & solve for $D_p = D_{ae}$

Notice → C depends on $D_p$ (or $D_{ae}$), so we must iterate

[This is an implicit eq.]

Simplest case → Stokes regime $C_D = 24/Re$

Ignore Cunningham

Ignore $\rho$ compared to $\rho_p$

Spherical particle of density $\rho_p \neq \rho_0$

$$\boxed{D_{ae} = D_p \sqrt{\frac{\rho_p}{\rho_0}}}$$

Eq. 3-1 in P&P book - but they never say how they derived it.