# LESSON 03B: REYNOLDS TRANSPORT THEOREM J. M. Cimbala

**In this lesson, we will**:

- Derive the **Reynolds Transport Theorem** (**RTT**) by application of **Leibniz Theorem**
- Discuss **material volumes** and how the RTT is a **link** between **Lagrangian** and **Eulerian** descriptions

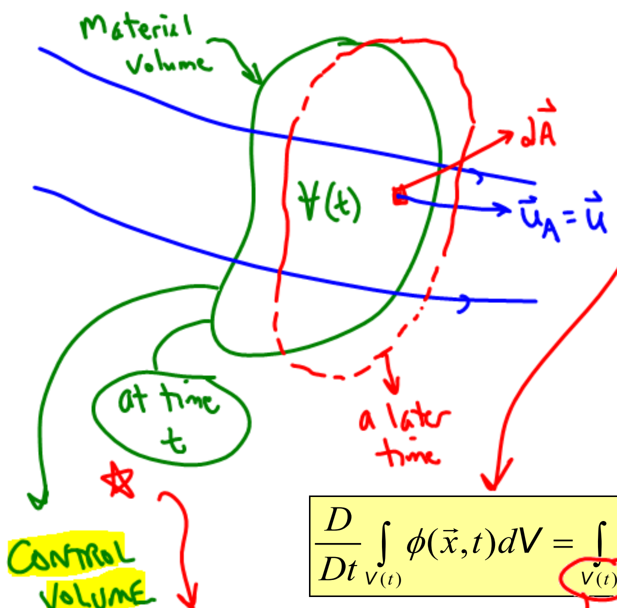
## Derivation of the Reynolds Transport Theorem

Start with the **3-D Leibniz Theorem** for function or property $\phi$, a function of $\vec{x}$ and $t$,

$\forall(t)$ moves independently of the fluid

$$\boxed{\frac{d}{dt}\int_{V(t)}\phi(\vec{x},t)dV = \int_{V(t)}\frac{\partial\phi(\vec{x},t)}{\partial t}dV + \oint_{A(t)}\phi(\vec{x},t)\vec{u}_A\cdot d\vec{A}}$$

For the special case of a ***material volume*** (a volume that moves with the fluid), $\boxed{\vec{u}_A=\vec{u}}$ ★




THIS MATERIAL VOLUME IS A SYSTEM [closed system]

ALWAYS CONTAINS THE SAME MASS

SINCE $\forall(t)$ IS A MATERIAL VOLUME, WE USE $\frac{D}{Dt}$ instead of $\frac{d}{dt}$

$$\boxed{\frac{D}{Dt}\int_{V(t)}\phi(\vec{x},t)dV = \int_{V(t)}\frac{\partial\phi(\vec{x},t)}{\partial t}dV + \oint_{A(t)}\phi(\vec{x},t)\vec{u}\cdot d\vec{A}} \quad (1)$$

But, ***at this instant in time***, since the control volume ***is*** the material volume (they are ***coincident***), we can set the volume as the control volume and the area as the control surface,

CV(t) CS(t)

$$\boxed{\frac{D}{Dt}\int_{V(t)}\phi(\vec{x},t)dV = \int_{CV(t)}\frac{\partial\phi(\vec{x},t)}{\partial t}dV + \oint_{CS(t)}\phi(\vec{x},t)\vec{u}\cdot d\vec{A}} \quad (2)$$

★ THIS IS THE R.T.T. !

LHS is the Material volume

RHS is the Control Volume

★ VALID AT ANY INSTANT IN TIME

$\vec{u}_A=\vec{u}$

WE CHOOSE $\forall(t)$ TO BE THE CV @ time t (material volume)

$\forall(t)$ & CV(t) are coincident @ time t

Summary: $\forall(t)$ = CV(t)

A(t) = CS(t)

## Summary of the Reynolds Transport Theorem

We applied Leibniz Theorem to the *special case* in which $V(t)$ is a ***material volume*** – *a volume that moves with the fluid*. As the fluid moves and distorts in the flow field, the material volume moves and distorts with it since, by definition, the material volume always contains the same physical mass of fluid. It follows that area $A$ also moves with the local fluid velocity. Thus, $\vec{u}_A = \vec{u}$ where $\vec{u}$ is the fluid velocity, a function of space and time in the Eulerian description. To stress that $V(t)$ is a ***material volume***, $D/Dt$ is used instead of $d/dt$ for time derivatives following material volumes.

The ***Reynolds Transport Theorem*** (***RTT***) for a *moving control volume* is thus

$\boxed{\dfrac{D}{Dt}\displaystyle\int_{V(t)} \phi(\vec{x},t)dV = \int_{CV(t)} \dfrac{\partial \phi(\vec{x},t)}{\partial t}dV + \oint_{CS(t)} \phi(\vec{x},t)\vec{u}\cdot d\vec{A}}$, where $\vec{u}$ is the ***absolute velocity of the fluid***, CV($t$) is the ***control volume***, and CS($t$) is the ***control surface***. In this general form of the Reynolds Transport Theorem, the control volume can be moving and distorting in any arbitrary fashion.

This is equivalent to $\boxed{\dfrac{D}{Dt}\displaystyle\int_{V(t)} \phi(\vec{x},t)dV = \dfrac{d}{dt}\int_{CV(t)} \phi(\vec{x},t)dV + \oint_{CS(t)} \phi(\vec{x},t)\vec{u}_{\text{relative}}\cdot d\vec{A}}$, where $\boxed{\vec{u}_{\text{relative}} = \vec{u} - \vec{u}_{CS}}$ is ***relative velocity***, the fluid velocity ***relative to the control surface***.

A simplification of the general Reynolds Transport Theorem is possible if the control volume is fixed in space. In such a case, the relative velocity of the fluid is identical to its absolute velocity. So, the ***Reynolds Transport Theorem*** for a ***fixed control volume*** is

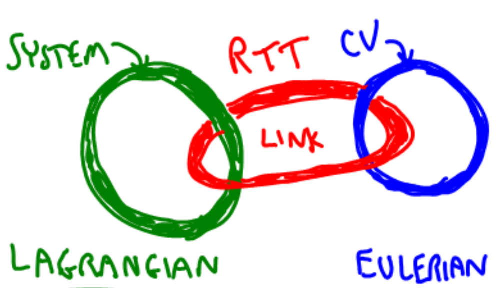
$\boxed{\dfrac{D}{Dt}\displaystyle\int_{V(t)} \phi(\vec{x},t)dV = \int_{CV} \dfrac{\partial \phi(\vec{x},t)}{\partial t}dV + \oint_{CS} \phi(\vec{x},t)\vec{u}\cdot d\vec{A}}$. Or, since the order of integration or differentiation does not matter, $\boxed{\dfrac{D}{Dt}\displaystyle\int_{V(t)} \phi(\vec{x},t)dV = \dfrac{d}{dt}\int_{CV} \phi(\vec{x},t)dV + \oint_{CS} \phi(\vec{x},t)\vec{u}\cdot d\vec{A}}$.

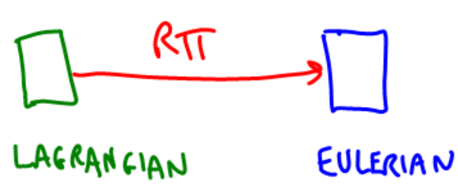
## Usefulness of the Reynolds Transport Theorem

The Reynolds Transport Theorem, in any of its forms above, contains a material volume on the left-hand side (LHS) and control volumes and control surfaces on the right-hand side (RHS). Thus, the LHS is in the ***Lagrangian*** or ***system*** frame of reference, while the RHS is in the ***Eulerian*** or ***control volume*** frame of reference.

***The usefulness of the Reynolds Transport Theorem is that it bridges the gap between the Lagrangian and Eulerian descriptions or frames of reference.***

It thus enables us to transform conservation laws (which apply directly to Lagrangian material volumes) into Eulerian forms, which are usually more desirable in fluid mechanics.

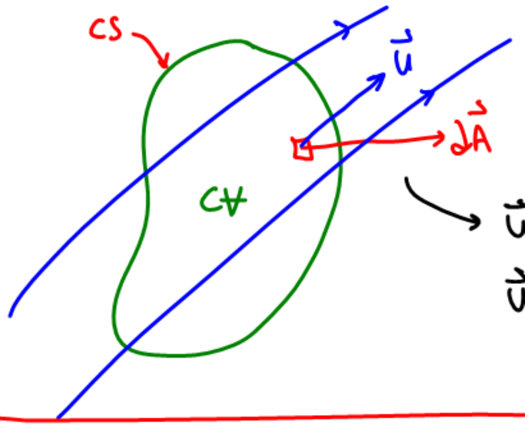


## The Right-Most Term in the Reynolds Transport Theorem

$$\frac{D}{Dt}\int_{V(t)} \phi(\vec{x},t)\,dV = \int_{CV(t)} \frac{\partial \phi(\vec{x},t)}{\partial t}\,dV + \oint_{CS(t)} \phi(\vec{x},t)\,\vec{u}\cdot d\vec{A}$$

$\vec{u}\cdot d\vec{A}$ is the volume flow rate of fluid crossing area element $dA$




$\vec{u}\cdot d\vec{A}$ is $\oplus$ if $\vec{u}$ is "out"

$\vec{u}\cdot d\vec{A}$ is $\ominus$ if $\vec{u}$ is "in"

$\boxed{\phi \text{ is typically a quantity per unit volume}}$ → WE LET $\boxed{\Phi = \phi\cdot\text{Volume}}$ ☆

☆ $\boxed{\oint_{CS} \phi\,\vec{u}\cdot d\vec{A} = \text{NET RATE OF } \underbrace{(\phi\cdot\text{Volume})}_{\Phi} \text{ FLOW } \underline{\underline{\text{OUT}}} \text{ OF THE CS}}$

E.G.,: Let $\phi = \rho = \dfrac{\text{mass}}{\text{Volume}}$ → $\underline{\Phi = \phi\cdot\text{volume} = \text{mass}}$

$\oint_{CS} \rho\,\vec{u}\cdot d\vec{A}$ = NET RATE OF MASS FLOW OUT OF THE CS

Dimensions

$\left\{\frac{m}{L^3}\,\frac{L}{t}\,L^2\right\} = \left\{\frac{m}{t}\right\}$ ✓

## Terms in the Reynolds Transport Theorem

$$\frac{D}{Dt}\int_{V(t)} \phi(\vec{x},t)\,dV = \int_{CV(t)} \frac{\partial \phi(\vec{x},t)}{\partial t}\,dV + \oint_{CS(t)} \phi(\vec{x},t)\,\vec{u}\cdot d\vec{A}$$

TOTAL RATE OF CHANGE OF $\Phi$ FOLLOWING A MATERIAL VOLUME

SYSTEM

↓

LAGRANGIAN

RATE OF CHANGE OF $\Phi$ DUE TO UNSTEADINESS OF $\phi$ WITHIN THE CV

RATE OF CHANGE OF $\Phi$ DUE TO NET OUTFLOW OF $\Phi$ THROUGH THE CS

CONTROL VOLUME

↓

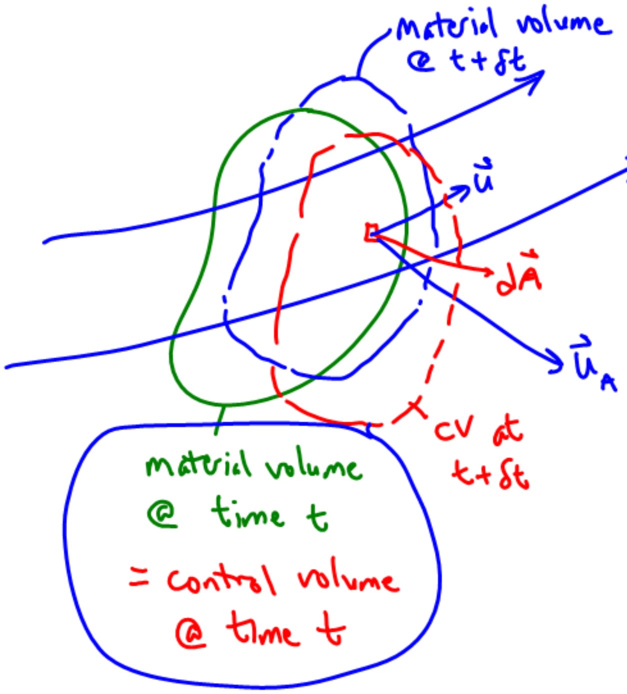
EULERIAN

CONSERVATION LAWS (FUNDAMENTALLY IN LAGRANGIAN FRAME) $\xrightarrow{\text{RTT}}$ EULERIAN

## Moving and/or Deforming Control Volume

$$\boxed{\frac{D}{Dt}\int_{V(t)}\phi(\vec{x},t)dV = \int_{CV(t)}\frac{\partial\phi(\vec{x},t)}{\partial t}dV + \oint_{CS(t)}\phi(\vec{x},t)\vec{u}\cdot d\vec{A}}\ \star$$

$\star$ GENERAL CASE → CV moves independently of the material volume
" " " " flow




Recall, 3-D Leibniz holds at any instant in time

∴ RTT holds at any instant in time

$\vec{u} \Rightarrow u_i$

$\vec{x} \Rightarrow x_i$ or $x_j$ or $x_k$

## Reynolds Transport Theorem in Tensor Notation

Start with the RTT in vector notation for property $\phi$, a function of $\vec{x}$ and $t$,

$$\boxed{\frac{D}{Dt}\int_{V(t)}\phi(\vec{x},t)dV = \int_{CV(t)}\frac{\partial\phi(\vec{x},t)}{\partial t}dV + \oint_{CS(t)}\phi(\vec{x},t)\vec{u}\cdot d\vec{A}}$$

In tensor notation $\phi = \phi(x_j,t)$, and the RTT becomes

$$\boxed{\frac{D}{Dt}\int_{V(t)}\phi(x_j,t)dV = \int_{CV(t)}\frac{\partial\phi(x_j,t)}{\partial t}dV + \oint_{CS(t)}\phi(x_j,t)u_i dA_i}$$