LESSON 03E: DIFFERENTIAL LINEAR MOMENTUM EQUATION   J. M. Cimbala

In this lesson, we will:

- Derive the **differential linear momentum equation** from its integral (control volume) form by application of **Gauss's theorem**
- Define and discuss **conservative** and **nonconservative** forms of differential equations of fluid motion
- Define and discuss **Cauchy's equation of fluid motion**

Derivation of the Differential Form of the Linear Momentum Equation

Start with the **control volume** (integral) form of the **linear momentum equation**,

$$\boxed{\int_{CV} \frac{\partial(\rho u_i)}{\partial t} dV + \oint_{CS} \rho u_i u_j dA_j = \int_{CV} \rho g_i dV + \oint_{CS} T_{ij} dA_j}$$

Apply **Gauss's theorem** to convert the two area integrals into volume integrals

$$\boxed{\int_V \frac{\partial \Phi}{\partial x_j} dV = \oint_A \Phi dA_j}$$

$$\oint_{cs} \underbrace{\rho u_i u_j}_{\Phi} dA_j = \int_{cv} \frac{\partial(\rho u_i u_j)}{\partial x_j} d\forall \qquad \oint_{cs} T_{ij} dA_j = \int_{cv} \frac{\partial}{\partial x_j}(T_{ij}) d\forall$$

$$\int_{cv} \left[\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} - \rho g_i - \frac{\partial}{\partial x_j}(T_{ij}) \right] d\forall = 0$$

For any arbitrary CV, $[\sim\sim] = 0$ so that $\int_{cv} [\sim] d\forall = 0$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial(\rho u_i u_j)}{\partial x_j} - \rho g_i - \frac{\partial}{\partial x_j}(T_{ij}) = 0$$

$$☆ \quad \boxed{\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \rho g_i + \frac{\partial}{\partial x_j}(T_{ij})}$$

DIFFERENTIAL LINEAR MOMENTUM EQ. ☆

☆ Vector equation

Final result: **Differential form of the linear momentum equation**:

$$\boxed{\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \rho g_i + \frac{\partial}{\partial x_j}(T_{ij})}$$

Conservative vs. Nonconservative Forms

Let's examine our differential equation for linear momentum from above,

$$\boxed{\frac{\partial}{\partial t}(\rho u_i)+\underbrace{\frac{\partial}{\partial x_j}(\rho u_i u_j)}=\rho g_i+\frac{\partial}{\partial x_j}(T_{ij})}$$

☆ CONSERVATIVE FORM

1) It was derived directly from the integral conservation form using Gauss's theorem

2) None of the variables being differentiated appears as a coefficient in another term where there are derivatives

$$\underbrace{\frac{\partial}{\partial x_j}(\rho u_i u_j)}_{\text{CONSERVATIVE}} = \underbrace{\rho u_j \frac{\partial u_i}{\partial x_j} + u_i \frac{\partial(\rho u_j)}{\partial x_j}}_{\text{NONCONSERVATIVE}}$$

Let Ψ be some property

$\boxed{\Psi \text{ is the conserved property}}$

$\psi = \Psi$ per unit mass

$\therefore \rho\psi = \Psi$ per unit volume

e.g. Ψ = mass

$\psi = 1$

$\rho\psi = \rho$

GENERAL CONSERVATIVE FORM OF THE DIFF. EQ. FOR CONS. OF Ψ:

☆ $$\boxed{\frac{\partial}{\partial t}(\rho\psi) + \frac{\partial}{\partial x_j}(\rho u_j \psi) = \text{SOURCES}}$$

ψ is the conserved property per unit mass

EXAMPLE: CONSERVATION OF MASS $\quad \psi = 1$

$$\boxed{\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x_j}(\rho u_j) = 0}$$

Diff. eq. for cons. of mass in conservative form ☆

MOMENTUM EQ.

Let Ψ = momentum = mass $\cdot u_i$

$\psi = u_i$

$$\boxed{\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \rho g_i + \frac{\partial}{\partial x_j} T_{ij}}$$

LINEAR MOMENTUM EQ.

CONSERVATIVE FORM

PRODUCT RULE

$$\rho \frac{\partial u_i}{\partial t} + u_i \frac{\partial \rho}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} + u_i \frac{\partial}{\partial x_j}(\rho u_j)$$

$$\rho \Bigg[\underbrace{\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}}_{= \frac{Du_i}{Dt}} \Bigg] + u_i \Bigg[\underbrace{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i)}_{= 0} \Bigg]$$

continuity

$$\boxed{\rho \frac{Du_i}{Dt} = \rho g_i + \frac{\partial}{\partial x_j} T_{ij}}$$

LINEAR MOMENTUM EQ. IN NONCONSERVATIVE FORM ☆

Cauchy's Equation

The differential linear momentum equation in nonconservative form is called ***Cauchy's equation***,

$$☆ \quad \boxed{\rho \frac{Du_i}{Dt} = \rho g_i + \frac{\partial}{\partial x_j}\left(T_{ij}\right)}$$

VALID FOR ANY FLUID (compressible or incompressible)

(Newtonian or Non-Newtonian)

& FOR SOLIDS!

How is T_{ij} related to primary variables in the flow?

(velocity, pressure, density, viscosity, ...)