In this lesson, we will:

- Derive the general **compressible Bernoulli equation** for inviscid regions of flow
- Simplify this compressible Bernoulli equation for **irrotational flow** and for **barotropic flow**

Derivation of the Compressible Bernoulli Equation for Inviscid Regions of Flow

Start with the **Euler equation** (inviscid form of the Navier-Stokes equation),

$$\boxed{\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+g_i}$$

- Use one of the vector equations from the previous lesson
- Use the "trick" from the previous lesson for g_i

$$\boxed{\cancel{\frac{\partial u_i}{\partial t}}+\frac{\partial}{\partial x_i}\left(\frac{1}{2}u_j u_j\right)+\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_i}(gz)=\cancel{u_j\varepsilon_{ijk}\omega_k}} \qquad (8)$$

EULER EQUATION

SIMPLIFICATIONS OF EQ. (8)

a. IRROTATIONAL, ISENTROPIC, STEADY, INVISCID, BUT COMPRESSIBLE

$\omega_k = 0$

- From thermo, $T\,ds = de + p\,dv$ specific volume $= \frac{1}{\rho}$

$$\boxed{\cancel{T\,ds} = dh - \frac{1}{\rho}dp}$$

$$\boxed{dh = \frac{1}{\rho}dp}$$

Following a fluid particle,

$$\therefore \boxed{\frac{Dh}{Dt}=\frac{1}{\rho}\frac{Dp}{Dt}} \rightarrow \underbrace{\cancel{\frac{\partial h}{\partial t}}}_{\text{steady}}+\cancel{u_i}\frac{\partial h}{\partial x_i}=\frac{1}{\rho}\underbrace{\cancel{\frac{\partial p}{\partial t}}}_{\text{steady}}+\frac{1}{\rho}\cancel{u_i}\frac{\partial p}{\partial x_i}$$

$$\boxed{\frac{\partial h}{\partial x_i}=\frac{1}{\rho}\frac{\partial p}{\partial x_i}}$$

EQ.(8) $\Rightarrow \quad \frac{\partial}{\partial x_i}\left(\frac{1}{2}\underbrace{u_j u_j}_{V^2}\right) + \frac{\partial h}{\partial x_i} + \frac{\partial}{\partial x_i}(gz) = 0$

$$\frac{\partial}{\partial x_i}\left[h + \frac{1}{2}V^2 + gz\right] = 0$$

$$\boxed{h + \frac{1}{2}V^2 + gz = \text{Constant everywhere}} \qquad (9)$$

BERNOULLI EQUATION FOR STEADY, ISENTROPIC, INVISCID, IRROTATIONAL FLOW
(but compressible)

[BELOVED BERNOULLI $\quad \frac{p}{\rho} + \frac{1}{2}V^2 + gz = \text{Constant everywhere}$]

$h = e + \frac{p}{\rho}$

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b. BAROTROPIC $\rightarrow$ $\rho = \rho(P)$ only

(e.g., Isentropic ideal gas

$\rho = \text{const}\cdot p^{\frac{1}{\gamma}}$

$\gamma = \frac{c_p}{c_v}$ (often γ is k instead))

1) Steady $\quad \boxed{\frac{1}{2}V^2 + \int\frac{dp}{\rho} + gz = \text{Constant along a streamline}}$

BERNOULLI EQUATION FOR BAROTROPIC, STEADY, INVISCID FLOW
(can be rotational)

Simplifications of the Compressible Inviscid Bernoulli Equation (CONTINUED)

2) Steady & Irrotational

$$\boxed{\frac{1}{2}V^2 + \int \frac{dp}{\rho} + gz = \text{const everywhere}}$$

BERNOULLI EQ FOR BAROTROPIC, STEADY, INVISCID, IRROTATIONAL FLOW

3) Unsteady & irrotational

$$\boxed{\frac{\partial \phi}{\partial t} + \frac{1}{2}V^2 + \int \frac{dp}{\rho} + gz = F(t)}$$

BERNOULLI EQ FOR BAROTROPIC, IRROTATIONAL, UNSTEADY, INVISCID FLOW

Summary: Various Forms of the Bernoulli Equation

Note: The fluid is assumed to be Newtonian, and gravity is assumed to act in the negative z direction [positive z is "up"]. All forms of the Bernoulli equation shown below are derived from the Navier-Stokes equation, along with some useful **vector identities**,

$$\boxed{u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_i}\left(\tfrac{1}{2}u_j u_j\right) - u_j \varepsilon_{ijk}\omega_k}, \text{ and } \boxed{\frac{\partial^2 u_i}{\partial x_j \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) - \varepsilon_{ijk}\frac{\partial \omega_k}{\partial x_j}}.$$

1. **Incompressible Flow**: Start with the ***incompressible N-S equation***:

$$\boxed{\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \rho g_i + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j}}$$

   a. **Incompressible, Unsteady, and Irrotational (can be viscous):**

$$\boxed{\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \tfrac{1}{2}u_i u_i + gz = F(t)} \text{ or } \boxed{\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \tfrac{1}{2}V^2 + gz = F(t)}$$

   where $\boxed{V^2 = u_i u_i = |\vec{u}|^2}$ = magnitude of the velocity vector squared, ϕ is the ***velocity potential*** defined by $\boxed{\vec{u} = \vec{\nabla}\phi}$ or $\boxed{u_i = \phi_{,i}}$ (ϕ is definable *only* if the flow is *irrotational*), and F is a function of time, but not of space.

   b. **Incompressible, Steady, and Irrotational (can be viscous):** [This is the "**Beloved Bernoulli Equation**."]

$$\boxed{\frac{p}{\rho} + \tfrac{1}{2}V^2 + gz = \text{constant everywhere}}$$

   c. **Incompressible, Steady, and Inviscid (can be rotational):** [This is the "**Second-Most Beloved Bernoulli Equation**."]

$$\boxed{\frac{p}{\rho} + \tfrac{1}{2}V^2 + gz = \text{constant along a streamline}}$$

2. **Compressible, Inviscid Flow**: Start with the ***compressible Euler equation***:

$$\boxed{\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + g_i}$$

   a. **Compressible, Inviscid, Steady, Isentropic, and Irrotational:**

$$\boxed{h + \tfrac{1}{2}V^2 + gz = \text{constant everywhere}}$$

   where h is the ***specific enthalpy*** $\boxed{h = e + p/\rho}$. [Note that this equation can also be obtained from the energy equation (first law of thermodynamics) when the same assumptions are made.]

b. Compressible but Barotropic, and Inviscid:

Note: ***Barotropic*** means that ***density is a function of pressure only***, i.e., $\boxed{\rho = \rho(p)}$.

1) Barotropic, Inviscid, and Steady (can be Rotational):

$$\boxed{\int \frac{dp}{\rho} + \tfrac{1}{2}V^2 + gz = \text{constant along a streamline}}$$

2) Barotropic, Inviscid, Steady, and Irrotational:

$$\boxed{\int \frac{dp}{\rho} + \tfrac{1}{2}V^2 + gz = \text{constant everywhere}}$$

3) Barotropic, Inviscid, Unsteady, and Irrotational:

$$\boxed{\frac{\partial \phi}{\partial t} + \int \frac{dp}{\rho} + \tfrac{1}{2}V^2 + gz = F(t)}$$

where velocity potential ϕ can be defined ***even if the flow is compressible***, as long as it is ***irrotational***.