In this lesson, we will:

- Define and discuss the **Boussinesq approximation** and how it modifies the Navier-Stokes equation
- Summarize the Boussinesq differential equations for **mass**, **momentum**, and **energy**

The Boussinesq Approximation

↳ "mildly compressible" or "nearly incompressible"

VALID FOR SMALL CHANGES IN TEMPERATURE

LET $\rho = \rho(T)$ in the gravity term, but constant everywhere else

- **LIQUIDS**: ALLOW FOR BUOYANCY DUE TO TEMPERATURE CHANGES
- **GASES**: ASSUME IDEAL GAS AT LOW MACH NUMBER
  ALLOW FOR BUOYANCY DUE TO TEMPERATURE CHANGES

$$\boxed{\rho = \rho_0\left[1 - \alpha\left(T - T_0\right)\right]}$$

★ BOUSSINESQ APPROXIMATION

where ρ_0 is a reference density @ $T = T_0$
T_0 is a reference temperature

$\Delta T = T - T_0$ must be small

(some authors use β) $\longrightarrow$ α = thermal expansion coefficient
= coefficient of thermal expansion

$$\boxed{\alpha = -\frac{1}{\rho}\left.\frac{\partial \rho}{\partial T}\right)_p}$$ ★ $$\{\alpha\} = \left\{\frac{1}{T}\right\}$$

α = constant in Boussinesq approximation

USE THIS EQ FOR DENSITY IN THE GRAVITY TERM

USE $\rho = \rho_0$ EVERYWHERE ELSE

APPROXIMATE μ, ν = constants

Summary: Boussinesq Approximate Equations for Mass, Momentum, and Energy

Note: The fluid is assumed to be Newtonian, and gravity is assumed to act in the negative z direction [positive z is "up"]. Also, $\nu = \mu/\rho_o$ is the (constant) kinematic viscosity.

Continuity: (same as the incompressible version):

$$\boxed{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0}$$

x*-momentum**: (x-component of the ***N-S equation, same as incompressible version):

$$\boxed{\frac{Du}{Dt}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho_o}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)}$$

y*-momentum**: (y-component of the ***N-S equation, same as incompressible version):

$$\boxed{\frac{Dv}{Dt}=\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho_o}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)}$$

z*-momentum**: (z-component of the ***N-S equation, same as incompressible version but with one extra source term):

$$\boxed{\frac{Dw}{Dt}=\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{1}{\rho_o}\frac{\partial p}{\partial z}-\left[1-\alpha\left(T-T_o\right)\right]g+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)}$$

Energy (heat equation): (same as incompressible version):

$$\boxed{\rho_o C_p\frac{DT}{Dt}=\rho_o C_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\cancel{\phi}}$$

Ignore

But the viscous dissipation term ϕ is usually negligibly small in the Boussinesq approximation, just as for incompressible flow.

(ϕ is same o.o.m. as other neglected terms)