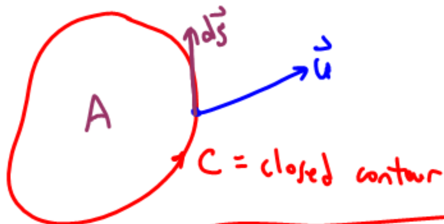
In this lesson, we will:

- Introduce **vorticity dynamics** and define **circulation**
- Define and discuss **Stokes' theorem** and apply it to circulation
- Define and discuss **Kelvin's circulation theorem** for a **material contour**
- Discuss the physical significance of Kelvin's circulation theorem

Circulation and Stokes' Theorem

CIRCULATION $= \boxed{\Gamma = \oint_C u_i\, ds_i \Leftrightarrow \oint_C \vec{u}\cdot d\vec{s}}$ ☆



STOKES' THEOREM

$$\boxed{\int_A \varepsilon_{ijk}\frac{\partial f_j}{\partial x_i}\, dA_k = \oint_C f_j\, ds_j \Leftrightarrow \int_A (\vec{\nabla}\times\vec{f})\cdot d\vec{A} = \oint_C \vec{f}\cdot d\vec{s}}$$ ☆

FLUID FLOW → LET $f_j = u_j$ $(\vec{f} = \vec{u})$

$$\int_A \underbrace{\varepsilon_{ijk}\frac{\partial u_j}{\partial x_i}}_{\omega_k} dA_k = \oint_C u_j\, ds_j = \Gamma \Rightarrow \boxed{\Gamma = \int_A \omega_k\, dA_k \quad \left(\Gamma = \int_A \vec{\omega}\cdot d\vec{A}\right)}$$

SOLID BODY ROTATION $\boxed{u_\theta = \frac{1}{2}\omega r}$

$\left[\Omega = \frac{1}{2}\omega\right]$

rotational but inviscid

$$\Gamma = \oint_C \vec{u}\cdot d\vec{s} = \int_0^{2\pi} \frac{1}{2}\omega r\, r\, d\theta = \pi r^2 \omega$$

LINE VORTEX $\boxed{u_\theta = \frac{\Gamma}{2\pi r}}$

irrotational (except @ origin)

& viscous stresses ≠ 0

$$\Gamma = \oint_C \vec{u}\cdot d\vec{s} = \int_0^{2\pi} \frac{\Gamma}{2\pi \cancel{r}}\cancel{r}\, d\theta = \Gamma$$

☆ KELVIN'S CIRCULATION THEOREM

IF CONTOUR C FOLLOWS THE FLUID, THEN Γ = CONSTANT

C is a material contour $\Rightarrow$ $\boxed{\dfrac{D\Gamma}{Dt} = 0}$ ☆ ☆

FOR AN "INVISCID", BAROTROPIC FLOW WITH CONSERVATIVE BODY FORCES

NO NET VISCOUS FORCES (VISCOUS STRESSES CAN BE NON-ZERO)

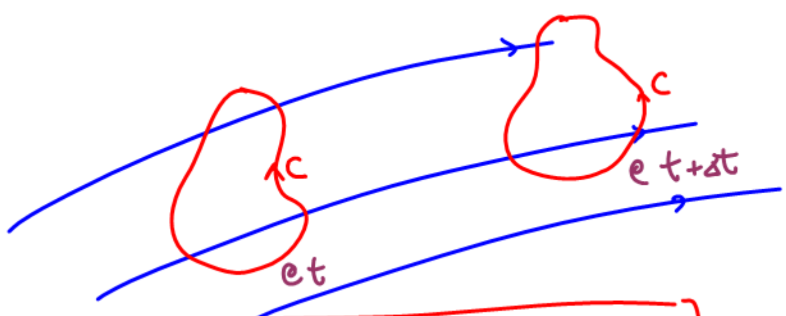
ρ = at most $\rho(p)$

INCOMPRESSIBLE IS A TYPE OF BAROTROPIC FLOW

CONSERVATIVE BODY FORCE CAN BE EXPRESSED AS THE GRADIENT OF A SCALAR

E.g., gravity $\quad \vec{g} = -\vec{\nabla}(gz)$ $\quad \uparrow z \quad \downarrow \vec{g}$

PHYSICAL SIGNIFICANCE OF KELVIN'S CIRCULATION THEOREM



$$\boxed{\Gamma_t = \Gamma_{t+\Delta t}}$$

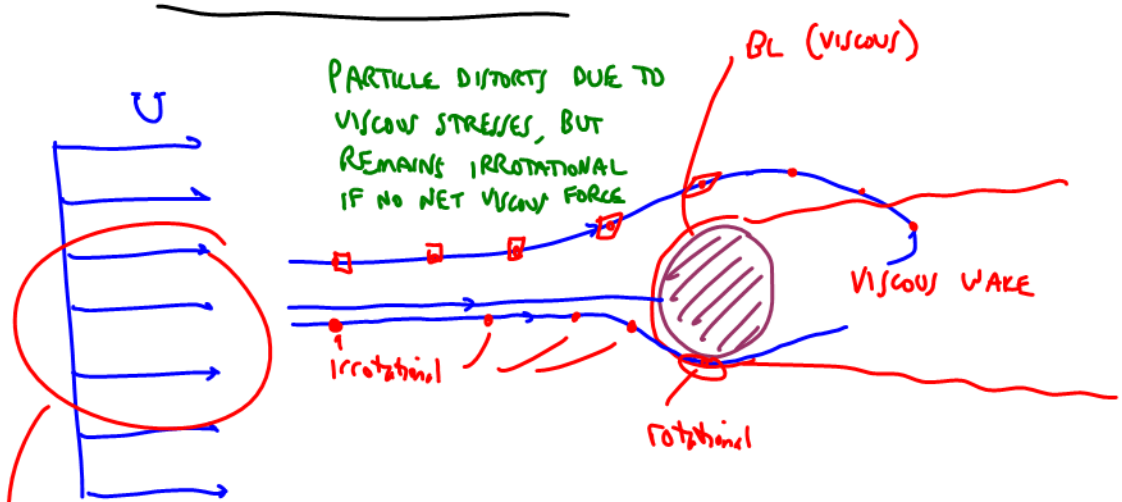
FOR AN IRROTATIONAL REGION OF FLOW $(\vec{\omega}=0) \rightarrow \underline{\underline{\Gamma = 0}}$

& shrink the material contour to a point (fluid particle)

☆ **A FLUID PARTICLE THAT IS IRROTATIONAL REMAINS IRROTATIONAL UNLESS IT ENCOUNTERS REGIONS WHERE NET VISCOUS FORCES ARE NON-ZERO**

DIRECT CONSEQUENCE OF K.C.T.

EXAMPLE: FLOW OVER AN OBJECT



$\Gamma = 0$ for any contour & $\vec{\omega} = 0$ for any fluid particle