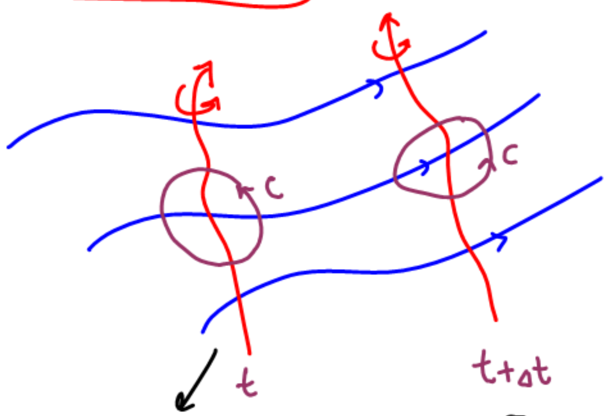
In this lesson, we will:

- State, explain, and in some cases prove four **Helmholtz vortex theorems** which apply to inviscid, barotropic flow with conservative body forces → Same as Kelvin's theorem
- Introduce **vortex tubes** and **vortex rings**
- Apply Helmholtz vortex theorems to a **viscously decaying line vortex**

Helmholtz Vortex Theorems

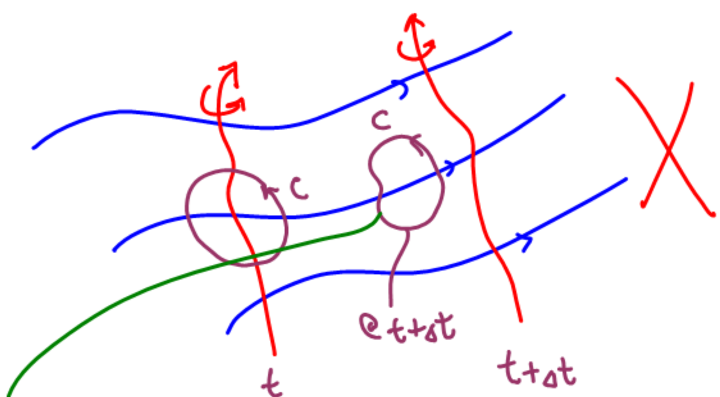
① Vortex lines move with the fluid

Vortex lines → Line of concentrated vorticity



Γ @ t=t Γ @ t+Δt must be the same

Kelvin → $\frac{D\Gamma}{Dt} = 0$ (Γ does not change following the fluid)



Γ here is not the same as Γ @ t=t

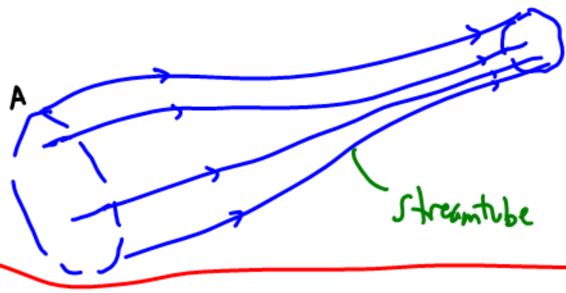
$\Gamma_{t+\Delta t} \neq \Gamma_t$ → THIS VIOLATES KELVIN'S CIRCULATION THEOREM!

∴ THE VORTEX LINE MUST MOVE WITH THE FLUID

(2) The strength of a vortex tube (i.e., its circulation) is constant along its length

ANALOGY: STREAMTUBE & INCOMPRESSIBLE CONS. OF MASS

↳ a group or bundle of streamlines

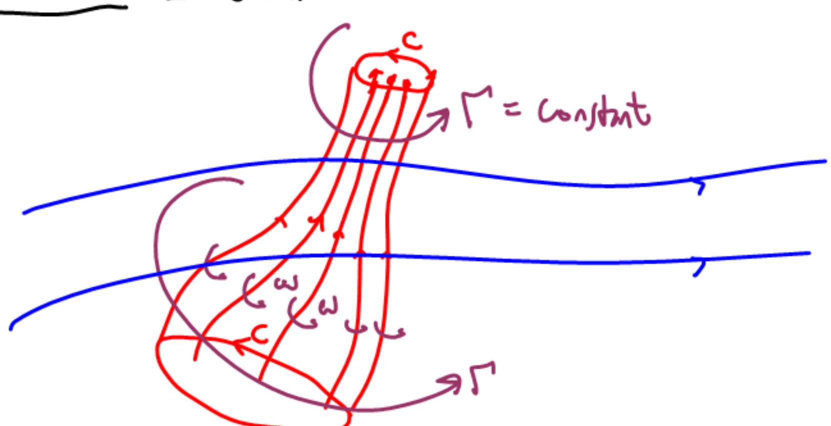


AS STREAMTUBE NARROWS, VELOCITY INCREASES

BUT, $\dot{\forall}$ remains constant

$$\dot{\forall} = \int_A \vec{u} \cdot d\vec{A} = \text{constant along a streamtube}$$ ☆

★ VORTEX TUBE ≡ GROUP OR BUNDLE OF VORTEX LINES



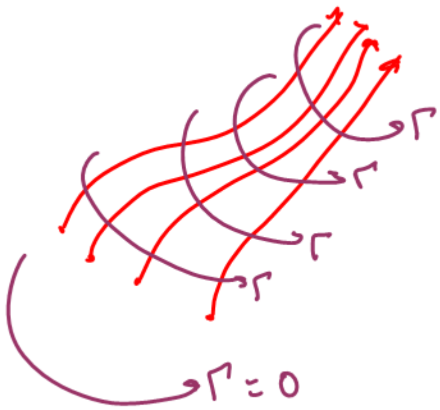
AS THE VORTEX TUBE NARROWS, ω ↑

$$\Gamma = \oint_C \vec{u} \cdot d\vec{s} = \int_A \vec{\omega} \cdot d\vec{A} = \text{constant along a vortex tube}$$

∴ CIRCULATION IS CONSTANT ALONG A VORTEX TUBE

③ A vortex tube cannot end in a fluid

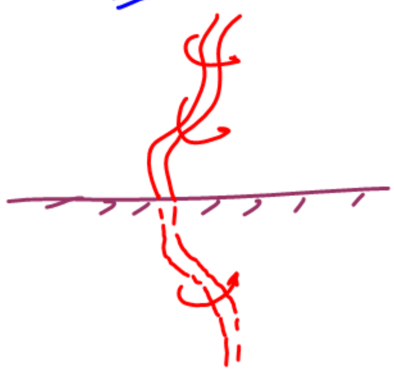
(follows directly from Helmholtz's second theorem)



JUST AS A STREAMTUBE CANNOT END IN A FLUID,

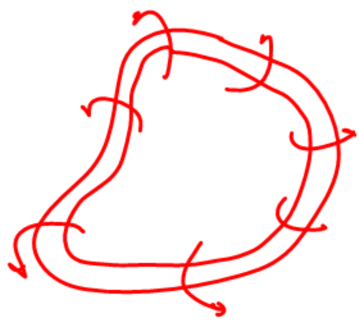
A VORTEX TUBE CANNOT END IN A FLUID

- A VORTEX TUBE CAN END AT A WALL



USE THE METHOD OF IMAGES

- A VORTEX RING



[We'll discuss vortex rings in the next lesson]

☆ A VORTEX TUBE CANNOT END IN A FLUID

(4) **THE STRENGTH (CIRCULATION) OF A VORTEX TUBE REMAINS CONSTANT IN TIME**

→ Follows directly from K.C.T.

$$\frac{D\Gamma}{Dt} = 0$$

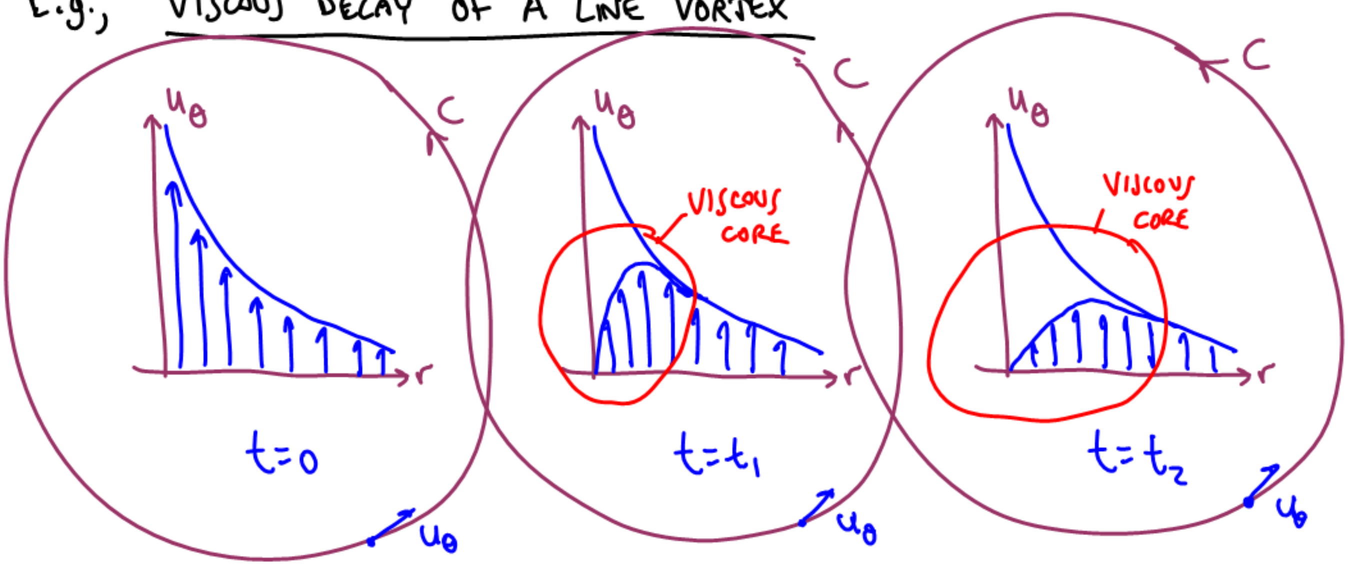
for a vortex line & for a vortex tube

[recall our restrictions: inviscid, barotropic flow with conservative body forces]

→ **CIRCULATION OF A VORTEX TUBE IS CONSTANT IN TIME**

- HELMHOLTZ'S FOURTH THEOREM STILL WORKS EVEN WITH NET VISCOUS FORCES AS LONG AS THE VISCOUS FORCES ARE INSIDE OUR CONTOUR C

E.g., <u>VISCOUS DECAY OF A LINE VORTEX</u>



$$\Gamma = \oint \vec{u}\cdot d\vec{s} \qquad \Gamma = \oint \vec{u}\cdot d\vec{s} \qquad \Gamma = \oint \vec{u}\cdot d\vec{s}$$

$$\Gamma @ t=0 \quad = \quad \Gamma @ t=t_1 \quad = \quad \Gamma @ t=t_2$$

★ Γ is a constant as this vortex decays in time provided that C is <u>outside</u> the viscous core