LESSON 06A: DERIVATION OF THE VORTICITY EQUATION — J. M. Cimbala

In this lesson, we will:

- Derive the **vorticity equation** in tensor notation
- Review some **vector identities** and the **epsilon-delta relation**

Derivation of the Vorticity Equation in Tensor Notation

Start with the incompressible Navier-Stokes equation with constant properties:

$$\boxed{\rho\frac{\partial u_i}{\partial t} + \rho u_j\frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \rho g_i + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j}}$$

$$u_j\frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_i}\left(\tfrac{1}{2}u_j u_j\right) - u_j\,\varepsilon_{ijn}\,\omega_n$$

→ move to RHS

$$\rho g_i = -\rho\frac{\partial}{\partial x_i}(gz)$$

$$\boxed{\rho\frac{\partial u_i}{\partial t} + \rho\frac{\partial}{\partial x_i}\left(\tfrac{1}{2}u_j u_j\right) = -\frac{\partial p}{\partial x_i} - \rho\frac{\partial}{\partial x_i}(gz) + \rho u_j\,\varepsilon_{ijn}\,\omega_n + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j}} \quad (1)$$

TAKE CURL OF EQ.(1) $\quad \vec{\nabla}\times\left[\text{Eq.}(1)\right]$

$$\Updownarrow$$

$$\varepsilon_{kmi}\frac{\partial}{\partial x_m}\left[\text{Eq.}(1)\right] \Rightarrow \text{VECTOR EQ WITH FREE INDEX } k$$

$$\left(\frac{\partial}{\partial x_m}\frac{\partial}{\partial t} = \frac{\partial}{\partial t}\frac{\partial}{\partial x_m}\right)$$

$$\boxed{\begin{aligned}&\rho\left[\overset{①}{\varepsilon_{kmi}\frac{\partial}{\partial t}\frac{\partial u_i}{\partial x_m}} + \overset{②}{\varepsilon_{kmi}\frac{\partial}{\partial x_m}\frac{\partial}{\partial x_i}\left(\tfrac{1}{2}u_j u_j\right)} - \overset{③}{\varepsilon_{kmi}\frac{\partial}{\partial x_m}\left(u_j\,\varepsilon_{ijn}\,\omega_n\right)}\right] \\ &= \overset{④}{-\varepsilon_{kmi}\frac{\partial}{\partial x_m}\frac{\partial}{\partial x_i}p} \overset{⑤}{- \rho\,\varepsilon_{kmi}\frac{\partial^2(gz)}{\partial x_m \partial x_i}} + \overset{⑥}{\mu\,\varepsilon_{kmi}\frac{\partial^3 u_i}{\partial x_m \partial x_j \partial x_j}}\end{aligned}} \quad (2)$$

• TERM ①

$$\rho\frac{\partial}{\partial t}\Big(\underbrace{\varepsilon_{kmi}\frac{\partial u_i}{\partial x_m}}_{\omega_k}\Big) \rightarrow \boxed{① = \rho\frac{\partial \omega_k}{\partial t}}$$

• TERM ②

$$\rho \, \varepsilon_{kmi} \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_i} \left(\tfrac{1}{2} u_j u_j \right)$$

ANTISYMMETRIC IN m & i

SYMMETRIC IN m & i

RECALL, THE CONTRACTED PRODUCT OF A SYMMETRIC & ANTISYMMETRIC TENSOR IS ZERO

$$\therefore \boxed{② = 0}$$

• TERM ④ → SAME ARGUMENT AS FOR TERM ② → $\boxed{④ = 0}$

• TERM ⑤ ' ' ' ' ' ' $\boxed{⑤ = 0}$

• TERM ⑥

$$\mu \, \varepsilon_{kmi} \frac{\partial^3}{\partial x_m \partial x_j \partial x_j} u_i = \mu \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} \left(\underbrace{\varepsilon_{kmi} \frac{\partial u_i}{\partial x_m}}_{\omega_k \; !} \right)$$

$$\boxed{⑥ = \mu \frac{\partial^2 \omega_k}{\partial x_j \partial x_j}}$$

• TERM ③

$$-\rho \, \varepsilon_{kmi} \frac{\partial}{\partial x_m} \left(u_j \varepsilon_{ijn} \omega_n \right) = -\rho \, \varepsilon_{kmi} \varepsilon_{ijn} \frac{\partial}{\partial x_m} \left(u_j \omega_n \right)$$

Recall the ε-δ relation

$$\varepsilon_{ijk} \varepsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

CHANGE ALL i to k

CHANGE ALL j to m

&

$$\Downarrow$$

$$\varepsilon_{kmi} \varepsilon_{ijn} = \delta_{kj} \delta_{mn} - \delta_{kn} \delta_{mj}$$

$$③ = -\rho \left[\delta_{kj} \delta_{mn} \frac{\partial}{\partial x_m} (u_j \omega_n) - \delta_{kn} \delta_{mj} \frac{\partial}{\partial x_m} (u_j \omega_n) \right]$$

contract

non-zero only if $j=k$

non-zero only if $j=m$

$$③ = -\rho\left[\delta_{mn}\frac{\partial}{\partial x_m}(u_k\omega_n) - \delta_{kn}\frac{\partial}{\partial x_j}(u_j\omega_n)\right]$$

$$③ = -\rho\left[\frac{\partial}{\partial x_n}(u_k\omega_n) - \frac{\partial}{\partial x_j}(u_j\omega_k)\right]$$

$$③ = -\rho\left[\frac{\partial}{\partial x_j}(u_k\omega_j) - \frac{\partial}{\partial x_j}(u_j\omega_k)\right]$$

USE PRODUCT RULE ON BOTH TERMS

$$③ = -\rho\left[\cancel{u_k\frac{\partial \omega_j}{\partial x_j}} + \omega_j\frac{\partial u_k}{\partial x_j} - \omega_k\cancel{\frac{\partial u_j}{\partial x_j}} - u_j\frac{\partial \omega_k}{\partial x_j}\right]$$

0 for incompressible flow

$$u_k\frac{\partial}{\partial x_j}\left(\varepsilon_{imj}\frac{\partial u_m}{\partial x_i}\right) = 0$$

$$\boxed{③ = -\rho\,\omega_j\frac{\partial u_k}{\partial x_j} + \rho\,u_j\frac{\partial \omega_k}{\partial x_j}}$$

$\mu = \rho\nu$

(2) becomes
$$\rho\left[\underbrace{\frac{\partial \omega_k}{\partial t} + u_j\frac{\partial \omega_k}{\partial x_j}}_{=\frac{D\omega_k}{Dt}}\right] = \rho\,\omega_j\frac{\partial u_k}{\partial x_j} + \mu\frac{\partial^2 \omega_k}{\partial x_j\partial x_j}$$

$$\boxed{\frac{D\omega_k}{Dt} = \omega_j\frac{\partial u_k}{\partial x_j} + \nu\frac{\partial^2 \omega_k}{\partial x_j\partial x_j}} \Longrightarrow \boxed{\frac{D\vec{\omega}}{Dt} = (\vec{\omega}\cdot\vec{\nabla})\vec{u} + \nu\nabla^2\vec{\omega}}$$

VORTICITY EQUATION FOR INCOMPRESSIBLE, NEWTONIAN FLOW

(with constant properties)