Equations for the Conservation of Energy
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Mechanical Energy Equation

Non-conservative form:

\[ \rho \frac{D}{Dt} \left( \frac{1}{2} u_i u_i \right) = \rho u_i g_i + u_i \frac{\partial \tau_{ij}}{\partial x_j} \]

Conservative form:

\[ \frac{\partial}{\partial t} \left( \frac{1}{2} \rho u_i u_i \right) + \frac{\partial}{\partial x_j} \left( \rho u_j u_i \right) = \frac{\partial}{\partial x_j} \left( \rho g_i + u_i \frac{\partial \tau_{ij}}{\partial x_j} \right) \]

or

\[ \frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j} \left( u_j E \right) = \rho u_i g_i + u_i \frac{\partial \tau_{ij}}{\partial x_j} \]

where \( \frac{1}{2} m V^2 \) is the kinetic energy (the conserved quantity), \( E = \frac{1}{2} \rho u_i u_i \) is the kinetic energy per unit volume, and \( \frac{1}{2} u_i u_i \) is the kinetic energy per unit mass. The terms on the right are sources (or sinks) of kinetic energy per unit volume.

Alternate conservative form:

\[ \frac{\partial}{\partial t} \left( \frac{1}{2} \rho u_i u_i \right) + \frac{\partial}{\partial x_j} \left( \rho u_j u_i \right) = \frac{\partial}{\partial x_j} \left( \rho g_i + u_i \frac{\partial \tau_{ij}}{\partial x_j} \right) \]

Thermal Energy Equation (Heat Equation – the “real” energy equation)

First Law of Thermodynamics for a Material Volume:

\[ \frac{D}{Dt} \int_V \rho (e + \frac{1}{2} u_i u_i) \, dV = \int_V \rho g_i u_i \, dV + \oint_A \tau_{ij} u_i \, dA - \oint_A q_i \, dA \]

where \( e \) = internal energy per unit mass (what most Thermodynamics books call \( u \)). Or, using the RTT,

First Law of Thermodynamics for a Control Volume:

\[ \int_{CV} \frac{\partial}{\partial t} \left[ \rho (e + \frac{1}{2} u_i u_i) \right] \, dV + \oint_{CS} \rho (e + \frac{1}{2} u_i u_i) u_j \, dA_j = \int_{CV} \rho g_i u_i \, dV + \oint_{CS} \tau_{ij} u_i \, dA_j - \oint_{CS} q_i \, dA_j \]

First Law of Thermodynamics in Differential form:

\[ \rho \frac{D}{Dt} \left( e + \frac{1}{2} u_i u_i \right) = \rho u_i g_i + \frac{\partial}{\partial x_j} \left( \tau_{ij} u_i \right) - \frac{\partial q_i}{\partial x_i} \]

Or, combining this with the mechanical energy equation, after some algebra we get an alternate form:

Differential Thermal Energy Equation (Heat Equation):

\[ \rho \frac{D}{Dt} \left( e + \frac{1}{2} u_i u_i \right) = - \frac{\partial q_i}{\partial x_i} + p \frac{\partial u_i}{\partial x_i} + \phi \]

where the terms are explained below:

I Rate of increase of internal energy of a fluid element per unit volume (following the fluid element). This term can be positive or negative.

II Rate of heat flux into the fluid element per unit volume (negative because \( \bar{q} \) is defined as positive outward). This term can be positive or negative.

III Rate of increase of internal energy per unit volume due to volumetric compression (negative because \( u_{i,i} = \) volumetric expansion, which is defined as positive, but compression increases the internal energy). This term can be positive or negative.

IV Rate of increase of internal energy per unit volume due to viscous dissipation. This term is always positive.