Leibniz Theorem and the Reynolds Transport Theorem for Control Volumes

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1-D Leibniz Theorem

The one-dimensional form of the ***Leibniz theorem*** allows us to differentiate an integral in which both the integrand and the *limits of integration* are functions of the variable with which the integral is being differentiated:

$$\boxed{\frac{d}{dt}\int_{x=a(t)}^{x=b(t)} F(x,t)dx = \int_{x=a(t)}^{x=b(t)} \frac{\partial F}{\partial t}dx + \frac{db}{dt}F(b,t) - \frac{da}{dt}F(a,t)}$$

Example: Find $\frac{d}{dt}\int_{x=0}^{x=ct} e^{-x^2}dx$. This integral cannot be solved in closed form and then differentiated. However, with Leibniz rule, the solution is easily found. The above expression reduces to $\boxed{ce^{-c^2t^2}}$ (to be done in class).

3-D Leibniz Theorem

The one-dimensional Leibniz theorem can be extended to three dimensions (volume and area integrals) as follows:

$$\boxed{\frac{d}{dt}\int_{V(t)} F(\vec{x},t)dV = \int_{V(t)} \frac{\partial F(\vec{x},t)}{\partial t}dV + \oint_{A(t)} F(\vec{x},t)\vec{u}_A \cdot d\vec{A}}$$ where

- $V(t)$ is some arbitrary volume, which may be changing with time, but *not necessarily moving with the fluid*.
- $A(t)$ is the surface (area) enclosing volume $V(t)$; A is also a function of time since *A moves with the volume*.
- $d\vec{A}$ is the outward normal vector of a little element of surface area on A.
- $F(\vec{x},t)$ is any fluid property (scalar, vector, or tensor of any order). *F is a function of space and time, independent of what the volume is doing* – it is a property of the fluid regardless of what we choose as volume $V(t)$.
- $\vec{u}_A$ is the velocity vector defining the motion of surface A. *This velocity is not necessarily the same as the velocity of the fluid itself*, but in general is a function of space and time in an Eulerian frame of reference.

Reynolds Transport Theorem

Now consider the *special case* where $V(t)$ is a ***material volume*** – *a volume that moves with the fluid*. As the fluid moves and distorts in the flowfield, the material volume moves and distorts with it, since by definition the material volume always contains the same physical mass of fluid at all times. It follows that area A also moves with the local fluid velocity; thus $\vec{u}_A = \vec{u}$ where $\vec{u}$ is the fluid velocity, a function of space and time in the Eulerian description. To stress that $V(t)$ is a material volume, D/Dt shall be used instead of d/dt for time derivatives following material volumes.

The ***Reynolds Transport Theorem*** (***RTT***) for a *moving control volume* is thus

$$\boxed{\frac{D}{Dt}\int_{V(t)} F(\vec{x},t)dV = \int_{\mathrm{CV}(t)} \frac{\partial F(\vec{x},t)}{\partial t}dV + \oint_{\mathrm{CS}(t)} F(\vec{x},t)\vec{u} \cdot d\vec{A}}$$

, where $\vec{u}$ is the ***absolute velocity***, CV(t) is the ***control volume***, and CS(t) is the ***control surface***. In this general form of the Reynolds Transport Theorem, the control volume can be moving and distorting in any arbitrary fashion. This is equivalent to

$$\boxed{\frac{D}{Dt}\int_{V(t)} F(\vec{x},t)dV = \frac{d}{dt}\int_{\mathrm{CV}(t)} F(\vec{x},t)dV + \oint_{\mathrm{CS}(t)} F(\vec{x},t)\vec{u}_{\mathrm{relative}} \cdot d\vec{A}},$$

where $\boxed{\vec{u}_{\mathrm{relative}} = \vec{u} - \vec{u}_{\mathrm{CS}}}$ is the velocity relative to the control surface.

A simplification of the general Reynolds Transport Theorem is possible if the control volume is fixed in space. In such a case, the relative velocity of the fluid is identical to its absolute velocity. So, the ***Reynolds Transport Theorem*** for a *fixed control volume* is

$$\boxed{\frac{D}{Dt}\int_{V(t)} F(\vec{x},t)dV = \int_{\mathrm{CV}} \frac{\partial F(\vec{x},t)}{\partial t}dV + \oint_{\mathrm{CS}} F(\vec{x},t)\vec{u} \cdot d\vec{A}}$$ or

$$\boxed{\frac{D}{Dt}\int_{V(t)} F(\vec{x},t)dV = \frac{d}{dt}\int_{\mathrm{CV}} F(\vec{x},t)dV + \oint_{\mathrm{CS}} F(\vec{x},t)\vec{u} \cdot d\vec{A}}.$$

Usefulness of the Reynolds Transport Theorem

The Reynolds Transport Theorem, in any of its forms above, contains a material volume on the left hand side (LHS) and control volumes and control surfaces on the right hand side (RHS). Thus, the LHS is in the ***Lagrangian*** or ***system*** frame, while the RHS is in the ***Eulerian*** or ***control volume*** frame. *The usefulness of the Reynolds Transport Theorem is that it bridges the gap between the Lagrangian and Eulerian descriptions or frames of reference*. It thus enables us to transform conservation laws (which apply directly to Lagrangian material volumes) into Eulerian forms, which are usually more desirable in fluid mechanics.