Today, we will:

- Continue our discussion of eigenvalues, eigenvectors, and tensor invariants
- Discuss the Gauss theorem (divergence theorem) in tensor notation
- Begin our discussion of Kinematics (Chapter 3 in the text) – Lagrangian vs. Eulerian descriptions, the material derivative, motion of fluid particles

10) Tensor Invariants

\[ I_1, I_2, I_3 \star \]  
[See HW 1 3.pdf "handout" on web for definitions]  
[Do not change with any rotation]

Analogies to vectors: When we rotate axes, the vector itself does not change. So, we can say that the magnitude of the vector is an invariant.

Similarly, there are 3 tensor invariants that do not change when we rotate the axes.

11) Gauss' Theorem

[Also called divergence theorem]

Recall,

\[ \oint V \cdot d\mathbf{A} = \iint \nabla \cdot Q \cdot dA \]

\[ dA = \text{outward normal area element} \]

In tensor notation we write Gauss' Theorem as

\[ \oint Q_i \cdot d\mathbf{A} = \iint Q_i \cdot dA_i \]

But, the beauty is that Q is not restricted to being a vector!

Q can be anything (scalar, vector, tensor)

(See next page)
Gauss's Theorem in tensor notation

\[ \int_{\Omega} \frac{\partial Q_i}{\partial x_i} \, d\Omega = \oint_{\partial \Omega} Q_i \, dA_i \]
for vector \( Q_i \)

This form applies for \( Q = \text{anything} \) (scalar, vector, or tensor)

General Gauss Theorem:

\[ \int_{\Omega} \frac{\partial Q}{\partial x_i} \, d\Omega = \oint_{\partial \Omega} Q \, dA_i \]

Relationship between principal axes & eigenvalues

Consider \( e_{ij} \) or \( \varepsilon \) as a symmetric second-order tensor

Definition of eigenvalue (how to calculate them):

\[ \begin{vmatrix} \varepsilon_{ii} - \lambda \end{vmatrix} = 0 \]

\[ \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \]

or,

\[ \begin{vmatrix} e_{ij} - \lambda \delta_{ij} \end{vmatrix} = 0 \]

\( \rightarrow \) Solve for \( \lambda \)

(get 3 roots, some of which may be repeated)

The \( \lambda \)'s are the eigenvalues.
Eigenvector

\[ \vec{e}_i \cdot \vec{b} = \lambda \vec{b} \] or \[ e_{ij} \vec{b}_j = \lambda \vec{b}_i \]

in general, \(3\) sets of these since there are \(3\) eigenvectors \(\lambda\) eigenvalues

\[ \begin{align*}
\hat{e}_1, \hat{e}_2, \hat{e}_3
\end{align*} \]

1) The eigenvalues of \(e_{ij}\) are the principal strain rates

2) The eigenvector of \(e_{ij}\) are the principal axes.

II. Kinematics (Chap. 3)

A. Intro "Kinematics" = "study of motion"

describe motion of fluids w/o discussing the physics

Two descriptions of fluid flow

1. Lagrangian \(\rightarrow\) follow individual particle

   e.g. billiard balls

   \[ \begin{align*}
   & \quad \vec{a} \\
   \end{align*} \]

   usually not convenient for fluid mechanics

2. Eulerian (usually preferred)

   set at a point in space \(i\), watch what flows by

   express quantities as a function of space \& time

   e.g. \[ \vec{u} = \vec{u}(\vec{x}, t) \]
\[ U_i = U_i(x_i, t) \]

*i is not a dummy index*

3. **Material Derivative** also called **substantial derivative**

   - **total**
   - **particle**

*Problem:*
- Eulerian is usually preferred
- But - the law of motion (e.g. conservation laws) are written in Lagrangian form.

  e.g. Newton's 2nd law \[ \mathbf{F} = m \ddot{\mathbf{r}} \] - written for objects like billiard balls, etc.

*Goal:*

  - Write Lagrangian laws in Eulerian form

  *How? → Use material derivative*

Let \( Q \) = any quantity (scalar, vector, tensor)

- As a field variable \( Q = Q(x_i, t) \) [Eulerian - Leibniz]

- Following a fluid particle \( Q_p = Q_p(x_p(t), t) \) [Lagrangian]

  \( \cdot \) = particle

  \( x_p \) = position vector of the particle

Time rate of change of \( Q \) following a fluid particle

\[
\frac{dQ_p}{dt} = \frac{d}{dt} Q_p(x_p(t), t) = \frac{\partial Q_p}{\partial t} + \frac{\partial Q_p}{\partial x_p} \frac{dx_p}{dt}
\]

(1) \[ \frac{\partial}{\partial t} \] = partial

\[ \frac{dx_p}{dt} \] = total

**CHAIN RULE**
But, since we are following a fluid particle, \( \frac{dx_{pi}}{dt} = u_i \), \( \frac{dx_{pi}}{dt} = u_z \) etc.

\[
\frac{dx_{pi}}{dt} = u_i \tag{2}
\]

Plug (2) into (1):

\[
\frac{dQ}{dt} = \frac{dQ}{d\xi} + u_i \frac{dQ}{dx_{pi}} \tag{3}
\]

At any instant in time, consider the fluid particle that happens to occupy a point in space under consideration

So, at that point in space at that time,

\[
\begin{align*}
X_i &= x_{pi} \quad \text{&} \quad Q_i = Q \\
\end{align*}
\]

Eq. (3) becomes

\[
\frac{dQ}{dt} = \frac{dQ}{d\xi} + u_i \frac{dQ}{dx_i} \tag{4}
\]

Give this a special notation \( \frac{dQ}{dt} \)

Material Derivative of \( Q \) = \( \frac{dQ}{dt} = \frac{dQ}{d\xi} + u_i \frac{dQ}{dx_i} \)

E.g. Newton's 2nd law

\[
F_i = ma_i
\]

Fundamentally written in Lagrangian description
First, let's consider force per unit volume $f_i = \frac{F_i}{V}$, $\rho = \frac{m}{V}$.

Let $a_i = \frac{du}{dt}$ following a fluid particle.

$A_{pi} = \frac{du}{dt}$

$A_{pi} = \frac{Du}{dt}$

From this, we have

$\rho \frac{Du}{dt} = f_i$

or

$\rho \left[ \frac{du}{dt} + u_k \frac{du}{dx_k} \right] = f_i$

Now this is Newton's law in Eulerian form.

LHS of Navier-Stokes eq.

Left hand side

We will discuss the RHS later on.