Today, we will:

- Discuss how to describe the motion of fluid particles
- Discuss the strain rate tensor $e_{ij}$
- Discuss vorticity and rotationality vs. irrotationality
- If time, compare circular flows: solid body rotation vs. line vortex
- Do Candy Questions for Candy Friday

B. Motion of Fluid Particles (CH. 3)

- 4 types of possible motion of a fluid particle
  
  a) Translation  
  
  b) Dilatation (linear strain)  
  
  c) Shear strain  
  
  d) Rotation  

In fluid mechanics, we talk about rates of these motions:

$$\text{Rate} = \frac{d}{dt} (\text{motion})$$

1. Rate of Translation of a fluid particle = $U_c$

2. Linear strain rate (rate of dilatation)

  "Dilate" means to become larger  
  
  "Dilatation" means fractional increase in length  
  
  Also called linear strain

Consider a line segment $AB$ in a fluid flow
\[ \text{time } t \]

\[ \delta x_1 \rightarrow u_1 \rightarrow u + \frac{2u}{dx_1} \delta x_1 \]

\[ A \rightarrow B \]

\[ \text{time } t + dt \]

\[ \text{new length} \]

\[ A' \rightarrow B' \]

\[ u_1, dt \]

\[ \int \left[ u + \frac{2u}{dx_1} \delta x_1 \right] dt \]

\[ \text{Original length} \]

\[ \text{new length} \]

\[ \delta x_1 \]

\[ \text{Linear strain} = \frac{\text{new length} - \text{orig. length}}{\text{original length}} \]

\[ \delta x_1 - u_1 dt + \left[ u + \frac{2u}{dx_1} \delta x_1 \right] dt \]

\[ \text{Linear strain rate in the } x_1 \text{ direction} = \frac{1}{dt} \left( \frac{\delta x_1}{dx_1} \right) = \frac{2u_1}{dx_1} = u_{1,1} \]

Similarly:

- \( x_2 \)
  - \( u_{2,2} \)
- \( x_3 \)
  - \( u_{3,3} \)

a. Strain rate tensor

Define: \( \varepsilon_{ij} = \text{strain rate tensor} \)

\[ \varepsilon_{11} = u_{1,1} \]

\[ \varepsilon_{22} = u_{2,2} \]

\[ \varepsilon_{33} = u_{3,3} \]

\[ \left\{ \varepsilon_{ii} = u_{i,i} \right\} \text{ (i not summed)} \]

The diagonal or normal components of the strain rate tensor.
b. Volumetric Strain Rate (Bulk strain rate)

\[
\text{rate of volume increase of a fluid particle per unit volume} = \frac{\partial \varphi}{\partial t} = U_{,1} + U_{,2} + U_{,3} = \varepsilon_{,ii}
\]

3. Shear Strain Rate

\[
\frac{\partial x}{\partial t} + \frac{\partial x}{\partial t}
\]

\( (x_1, x_2 \text{ plane}) \)

a. Define rate of shear strain in the 1, 2 plane = rate of decrease of the angle between two originally perpendicular lines (e.g. \( x_1 \) and \( x_2 \) directions) at point \( P \)

\[
= \frac{dx}{dt} + \frac{d\beta}{dt}
\]

See lab handout. After some trig

Shear strain rate \( r_{12} = \frac{du_1}{dx_2} + \frac{du_2}{dx_1} = U_{1,2} + U_{2,1} \)

Similarly

Shear strain rate \( r_{23} = \frac{du_2}{dx_3} + \frac{du_3}{dx_2} \)

\[
13 \quad \frac{du_1}{dx_3} + \frac{du_3}{dx_1}
\]
6. Strain Rate Tensor, $e_{ij}$

Combine normal (linear) and shear (tangential) strain rates into one strain rate tensor.

Define:
- $e_{11} = u_{1,1}$
- $e_{22} = u_{2,2}$
- $e_{33} = u_{3,3}$
- $e_{12} = \frac{1}{2} (u_{1,2} + u_{2,1})$
- $e_{13} = \frac{1}{2} (u_{1,3} + u_{3,1})$
- $e_{23} = \frac{1}{2} (u_{2,3} + u_{3,2})$

Factor of $\frac{1}{2}$ is necessary to combine these into one tensor.

$e_{ij} = \begin{bmatrix}
    e_{11} & e_{12} & e_{13} \\
    e_{21} & e_{22} & e_{23} \\
    e_{31} & e_{32} & e_{33}
\end{bmatrix}$

Diagonal components are the normal strain rates.

Off-diagonal components are half of the shear strain rates.

$e_{ij} = \frac{1}{2} (u_{ij,j} + u_{ij,i})$

(It is symmetric.)

4. Rate of rotation i.e., vorticity

Rate of rotation = avg. rotation rate of two originally 1 line.
Rate of rotation \( \dot{\beta} \) = rate of rotation about the \( x_3 \) axis

\[
\dot{\beta} = \frac{1}{2} \left( \frac{d\beta}{dt} - \frac{dx}{dt} \right)
\]

After some try:

Rate of rotation \( \dot{\beta} \) = \( \frac{1}{2} \left( \frac{d\beta}{dt} - \frac{dx}{dt} \right) \)

\[
= \text{angular velocity vector (component in } x_3 \text{ direction)}
\]

b. Vorticity

Vorticity vector \( \vec{\omega} \) = Twice the angular velocity vector

\[
\vec{\omega} = \nabla \times \vec{u}
\]

\[
\omega_3 = \frac{du_2}{dx_1} - \frac{du_1}{dx_2}
\]

Similarly for other 2 components

\[
\omega_k = \varepsilon_{ijk} \frac{du_j}{dx_i}
\]

\[
\omega_k = \varepsilon_{ijk} u_{j,i}
\]

[see pdf file on website for other components]