Today, we will:

- Do an example problem – Control Volume Example: Flat Plate Boundary Layer
- (See handout on website for the problem setup, as copied below)

### Given:
Consider flow over a two-dimensional flat plate. The freestream flow upstream of the plate is steady, uniform, and incompressible, with pressure $p_\infty$ and velocity $U_\infty$ in the $x$-direction as shown. A boundary layer of thickness $\delta$ grows along the top and bottom sides of the plate. (Only the top boundary layer is shown in the sketch.) The boundary layer velocity profile is shown at some $x$-location along the plate.

### To do:
Calculate the drag per unit depth, $D(x)/b$, on one side of the plate (for a plate of length $x$ and depth $b$).

### Solution:
First pick a control volume. There are several choices - let’s pick the simplest one, as sketched below. There are 6 surfaces which define the closed control surface. Namely, surface I is a vertical plane just upstream of the leading edge of the plate. Surface II is a vertical plane at location $x$. The plate itself is surface III. Surface IV is a curved sheet that follows a streamline outside of the boundary layer. Surface V is in the plane of the sketch, and surface VI is a plane parallel to surface V, but at some arbitrary distance $b$ away from plane V (towards the viewer, i.e. out of the page in the $z$-direction).

Notice that to conserve mass, the streamline outside the boundary layer must curve upward (in the $y$-direction) as sketched. Thus, surface II is slightly higher than surface I ($H_2 > H_1$).

Now apply the integral form of conservation of mass (Eq. 4.7):

$$0 = \oint_{cs} \rho \frac{du}{dt} dA$$

(incomp., steady)

The first term on the RHS drops out because in this problem the flow is steady and incompressible.

Split the second term on the RHS into its six parts, one for each surface I, II, III, IV, V, and VI, which together make up the entire closed control surface:

$$\oint_{cs} \rho u dA = \int_{I} \rho u dA + \int_{II} \rho u dA + \int_{III} \rho u dA + \int_{IV} \rho u dA + \int_{V} \rho u dA + \int_{VI} \rho u dA$$

The last two terms above are zero since the problem is two-dimensional (no mass flux into or out of the plane of the sketch).

...... Problem to be completed in class ......
Consider the other u surging separately

\[ u = (u_0, 0, 0) \]
\[ dA = (b \, dy, 0, 0) \]

\[ u_i \, dA_i = -U_0 \, b \, dy + O + O \]

\[ \int \rho u_i \, dA_i = b \rho \int_{-U_0} \, dy = -b \rho U_0 \, H \]

It is an inFlow

\[ \bar{u} = (u(y), v, 0) \]
\[ dA = (b \, dy, 0, 0) \]

\[ u_i \, dA_i = u \, b \, dy + v \frac{dy}{dx} + O \]

\[ \int \rho u_i \, dA_i = b \rho \int_{y = 0}^{H_2} u \, dy \]

Along the wall

\[ \bar{u} = (0, 0, 0) \]
\[ dA = (0, -b \, dx, 0) \]
\[ u_i \, dA_i = 0 \]

\[ \int_{\text{III}} = 0 \]

( streamline on top)

\[ \bar{u} \perp dA \implies \quad u_i \, dA_i = 0 \quad \text{(dot product of } 2 \perp \text{ vectors)} \]

"Clever" choice of control volume

\[ \int_{\text{IV}} = 0 \]
\[ M_{\text{nl}} \text{ eq becomes} \]
\[ 0 = -b \rho U_\infty H_1 + b \rho \int_{y=0}^{H_2} u \, dy \]
\[ \therefore U_\infty H_1 = \int_{y=0}^{H_2} u \, dy \quad (1) \quad \text{Result of ans. of} \ M_{\text{nl}} \]

Linear mom. eq. [we are concerned only with x-component (x, comp.)]

Let \( i = i \) in C.V. mom. eq.: (x, component)

\[ \int \frac{\partial}{\partial t} (\rho u_i) \, dV + \int \rho u_i u_j \, dA_j = \int \rho g_i \, dV + \int \zeta_{ij} \, dA_j \quad \text{steady} \]

let \( \vec{g} = (0, 0, -g) \) \(-z \text{ direction}\)

\[ U_\infty H_1 \]

\[ \int \rho u_i u_j \, dA_j = \int \zeta_{ij} \, dA_j \quad (2) \]

\[ \text{LHS} \quad \text{RHS} \]

\[ (2) \text{LHS} = \int_1 + \int_2 + \int_3 + \int_4 + \int_5 + \int_6 \rho u_i u_j \, dA_j \]

\[ I: \] \( \vec{u} = (U_\infty, 0, 0) \quad \int_1 \quad U_i U_j \, dA_j = U_\infty (-b \rho dy + o + o) \)

\[ d\vec{A} = (-by, 0, 0) \]

\[ \text{LHS}_1 = -b \rho U_\infty^2 \int_0^{H_1} dy = -b \rho U_\infty^2 H_1 \]
Recall our conv. of mass result (E3 (13))

Plug (1) into the above

\[
(2) \text{LHS}_x = -\rho b U_\infty \int_0^{H_2} u \, dy
\]

\[
\dot{u} = (u, v, 0) \quad \Rightarrow \quad u_i u_j \, dA_j = u \left[ u \, dy + 0 \cdot v + 0 \right] = u^2 \, dy
\]

\[
\text{LHS}_\Pi = \rho \int u_i u_j \, dA_j = \rho b \int_0^{H_2} u^2 \, dy
\]

\[
\text{LHS}_\Pi = 0
\]

\[
U_1 = 0 \quad \Rightarrow \quad \text{LHS}_\Pi = 0
\]

\[
U_j \, dA_j = 0 \quad \left(\text{unstressed} \right)
\]

Combine:

\[
(2) \text{LHS} = \rho b \int_0^{H_2} u^2 \, dy - \rho b U_\infty \int_0^{H_2} u \, dy
\]

\[
\text{RHS} = \oint \tau_{ij} \, dA_j = \int_1 + \int_2 + \int_3 + \int_4 + \int_5 + \int_6 \tau_{ij} \, dA_j
\]

\[
\tau_{ij} \, dA_j + \tau_{il} \, dA_l + \tau_{ij} \, dA_j
\]

Constitutive eq. for \( \tau_{ij} \) (incompressible fluid): \( \tau_{ij} = -\mu \delta_{ij} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \)
\[ Z_{ii} = -p + 2\mu \frac{du}{dx} \]
\[ Z_{ij} = \mu \left( \frac{du}{dy} + \frac{dv}{dx} \right) \]
\[ Z_{ij} = \mu \left( \frac{dv}{dy} + \frac{du}{dx} \right) = 0 \]

\[ (2) \quad \text{RHS}_I = \int_i \tau_{ij} dA_j \]
\[ \vec{u} = (U_{\infty}, 0, 0) \]
\[ dA = (0, b dx, 0) \]
\[ \tau_{ij} dA_j = \tau_{ii} (-b dy + 0 + 0) \]
\[ = \left[ -\rho_{\infty} + 2\mu \frac{du}{dx} \right] [-b dy] \]
\[ A_t \quad I \quad u = U_{\infty} = \text{constant} \]

\[ (2) \quad \text{RHS}_I = \int_0^{H_i} (-\rho_{\infty})(-b dy) = b \rho_{\infty} H_i \]

\[ \text{II} \]
\[ (2) \quad \text{RHS}_II \quad \quad dA = (b dy, 0, 0) \]
\[ \tau_{ij} dA_j = b \tau_{ii} dy \]
\[ \tau_{ii} = -\rho_{\infty} + 2\mu \frac{du}{dx} \]

BL approx. \( \rho = \rho_{\infty} = \text{constant} \) across the BL

\[ \frac{du}{dx} \quad v \quad \text{very small} \]

\[ (2) \quad \text{RHS}_II \quad = -b \rho_{\infty} H_i \]

\[ \text{III} \]
\[ (\text{on the plate}) \quad \quad \vec{u} = 0 \]
\[ dA = (0, -b dx, 0) \]
\[ \tau_{ij} dA_j = -\tau_{12} \frac{b dx}{dx} \]

Define \( \tau_w = \text{wall shear rate} = M \frac{dy}{dy} \quad \text{wall} \)
\[ \text{(2) } \text{RHS IV} = \int y \partial_{ij} A_i = \int (b \partial_{ij} y) + \int (b \partial_{ij} x) \]

\[ \partial_{ij} A_i = \frac{\partial y}{\partial x} (b \partial_{ij} y) + \frac{\partial x}{\partial y} (b \partial_{ij} x) \]

\[ \partial_{ij} A_i = -K_{\infty} + \frac{2 \mu}{d} \frac{\partial y}{\partial x} \]

\[ \text{Outside the BL } \quad \frac{u}{U_{\infty}} = \text{const} \]

\[ \text{Outside the BL } \quad \frac{u}{U_{\infty}} = \text{const} \]

\[ \text{(2) } \text{RHS IV} = b \rho_{\infty} (H_2 - H_1) \]

\[ \text{THE REST OF THIS WAS DONE AFTER CLASS: [ran out of time]} \]

\[ \text{ADD ALL THE TERMS TOGETHER:} \]

\[ \text{(2) } \text{RHS } = \int_{0}^{y} \partial_{ij} A_i = \int (b \rho_{\infty} H_1) - \int (b \rho_{\infty} H_2) - \int D + \int b \rho_{\infty} (H_2 - H_1) \]

\[ \text{Finally, (2) becomes} \]

\[ D = b \rho U_{\infty}^2 \int_{0}^{y} \frac{u}{U_{\infty}} (1 - \frac{u}{U_{\infty}}) dy \]