Today, we will:

- Finish the boundary layer problem we started last time
- Do an example problem – incompressible Couette flow
- Discuss conservation of energy
- Do Candy Questions for Candy Friday

Flat plate boundary layer problem (continued from last time):

When we substitute the LHS and RHS results into Eq. 2, we get

\[ D = b \rho U_\infty^2 \int_{y=0}^{\infty} \frac{u}{U_\infty} \left(1 - \frac{u}{U_\infty}\right) dy \]

\[ \theta = \int_0^\infty \frac{u}{U_\infty} \left(1 - \frac{u}{U_\infty}\right) dy \]

\[ \frac{D}{b} = \text{drag per unit depth on one side of the plate} \]

\[ \frac{D}{b} = \rho U_\infty^2 \theta \]
6. Example problems (continued)
   a. Flat plate boundary layer (this is a control volume/integral analysis)
   b. Incompressible Couette flow (this is a differential analysis)

Example

Given: A fluid flows between two infinite parallel flat plates as sketched. The following assumptions/approximations are made:

1. All flow field variables and fluid properties are independent of \( x \).
2. The flow is steady.
3. The flow is two-dimensional. in \( xy \) plane.
4. Gravity acts in the \(-y\) direction; \( \mathbf{g} = (0, -g, 0) \).

(a) To do: Calculate the \( y \)-component of velocity, \( v \).

Solution:

\[
\begin{align*}
\frac{\partial}{\partial x} ( \text{anything} ) &= 0 \\
\frac{\partial}{\partial t} ( \ldots ) &= 0 \\
\frac{\partial^2}{\partial x^2} ( \ldots ) &= 0 \\
g_1 &= 0, \quad g_2 = -g, \quad g_3 = 0
\end{align*}
\]

Continuity

\[
\frac{\partial u}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]

\[
\frac{\partial v}{\partial y} = 0 \quad \Rightarrow \quad v \neq \text{func. of } y
\]

But \( v \neq \text{func. of } x, t, z \) \( \quad \Rightarrow \quad v = \text{constant} \)

Apply a Boundary condition \( \text{(BC)} \) somewhere.

\( \text{at bottom wall, at } y = 0, \quad v = 0 \) \( \quad \text{(no slip)} \) \( \Rightarrow \quad V = 0 \) everywhere.

(b) Calc. pressure field, \( p \)

Solution: \( y \)-moment eq. \( (i = 2) \)

\[
p \frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} + \rho \mathbf{g} \cdot \mathbf{e}_i + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j}
\]

Set \( i = 2 \)
\[ \nabla \cdot \mathbf{u} = 0 \]

Continuity: \( v = 0 \) everywhere

\[ \frac{\partial p}{\partial y} = -\rho g \]

But, \( p \neq \text{func. of } x, t, \) or \( z \) \( \rightarrow \) \( p = p(y) \) only

Write \( \frac{\partial p}{\partial y} = -\rho g \)

Integrate \( p = -\rho gy + \text{const} \)

Example: Suppose \( p = p_w @ y = 0 \) (bottom wall)

\[ p = p_w - \rho gy \]

Hydrostatic Pressure

(c) Calculate \( u \)

Solution: Use x-mom eq. \((i=1)\)

\[ p \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \rho g_1 + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \]

\[ \begin{align*}
\text{(2)} & \quad \text{(1)} & \quad \text{(cont.)} & \quad \text{(3)} & \quad \text{(1)} & \quad \text{(1)} & \quad \text{(1)} \\
\text{(7)} & \quad \text{(9)} & \quad \text{(1)} & \quad \text{(3)} & \quad \text{(1)} & \quad \text{(1)} & \quad \text{(1)} & \quad \text{(1)} \\
\end{align*} \]

\( \text{since } \frac{\partial u}{\partial x} = 0 \) everywhere \( \frac{\partial^2 u}{\partial x^2} = \frac{1}{d_x} \left( \frac{d^2 u}{d x^2} \right) = \frac{1}{d_x} (0) = 0 \)

\[ \frac{\partial^2 u}{\partial x^2} = 0 \]

But, \( u \neq \text{func. of } x, t, z \)

\[ \therefore u = u(y) \text{ only} \]
Integrate twice \( \frac{du}{dy} = c_i \)

\[ u = c_1 y + c_2 \]

Apply BC to calc. \( c_1 \) & \( c_2 \):

\[ \begin{align*}
  & \text{No slip} \quad \begin{cases}
     & \text{at } y = 0, \ u = 0 \rightarrow c_2 = 0 \\
     & \text{at } y = b, \ u = U \rightarrow c_1 = \frac{U}{b}
  \end{cases} \\
  \end{align*} \]

\[ u = u(y) = \frac{U y}{b} \rightarrow \text{linear} \]

Compare to Bernoulli eq. \[ p + \frac{1}{2} \rho U^2 + \rho g y = \text{const.} \]

Plug in \( u = \frac{U y}{b} \)

\[ p = \text{const.} - \rho g y - \frac{1}{2} \rho \left( \frac{U}{b} \right)^2 y^2 \]

But our \( y \)-mom result gave \( p = \text{const.} - \rho g y \)

= linear in \( y \)

Bernoulli does not apply here everywhere, only along streamlines since this is a rotational flow.

Fluid partly rotate \[ \text{Fluid partly rotate} \]
E. CONSERVATION OF ENERGY

1. Notation
   [some of Kundu's notation is confusing]
   \[ E = \text{internal energy per unit mass} \quad (\text{most thermo books use } u) \]
   \[ E + \frac{u_i^2}{2} = \text{stored energy per unit mass} \]
   \[ E = \rho \frac{u_i^2}{2} = \text{kinetic energy per unit volume} \]
   \[ u_i^2 = u_i u_i \quad [i \text{ is a dummy index}] \]

   a. Intro
      . Eq. for the kinetic energy of the fluid
      . Derived from momentum & mass eqs
      . It is not the "real" energy eq [from 1st law of thermodynamics]
      . It is "unnecessary" in terms of our primary unknowns [6 unknowns + 6 eqs]
      Why we use it?
      . For convenience & physical understanding
      . Useful later on when discussing turbulence
      . We will simplify the "real" energy eq by combining with the mech. energy eq.
   b. Derivation
      . Start with Cauchy's eq.
      \[ \frac{\partial u_i}{\partial t} = \rho g_i + \frac{\partial c_{ij}}{\partial x_j} \]
Mult. by $u_i$ (taking dot product of $\ddot{u}_i$ & Cauchy eq.)

$\rho u_i \frac{Du_i}{Dt} = \rho u_i \dot{u}_i + u_i \frac{\partial \dot{u}_i}{\partial x_j}$

- Reverse product rule

$\rho \frac{D}{Dt} (\frac{1}{2} u_i u_i) = \rho u_i \dot{u}_i + u_i \frac{\partial \dot{u}_i}{\partial x_j}$

Recall Kunde defn $E = \frac{\rho u_i u_i}{2}$ k.e. per unit volume

$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j} (u_j E) = \rho u_i \dot{u}_i + u_i \frac{\partial \dot{u}_i}{\partial x_j}$

MECHANICAL ENERGY EQ (in conservative form)

LHS = total rate of change of $E$ following a fluid element

RHS = sources [sink] contributing to the change of $E$