Today, we will:

- Continue to discuss the interaction of vortices
- Discuss various time-marching schemes to predict the interaction of vortices
- Begin a discussion of the vorticity equation
- Do Candy Questions for Candy Friday

\[ r^2 = (x_2-x_1)^2 + (y_2-y_1)^2 \]

\[
U_i = \frac{\Gamma_i}{2\pi r} \sin \theta = \frac{\Gamma_i}{2\pi r} \frac{dy}{r} = \frac{\Gamma_i}{2\pi} \frac{dy}{r^2}
\]

Similarly, \[ V_i = -\frac{\Gamma_i}{2\pi} \frac{\Delta x}{r^2} \]

Influence of \( \Gamma_2 \) on \( \Gamma_1 \):

\[ (x, y, t) = \text{position of vortex } \Gamma_1 \text{ at time } t \]

\[ (x_0, y_0, t) = \ldots \]
Write as 4 1st-order nonlinear ode's: four variables $x_1, x_2, y_1, y_2$

\[ U_1 = \frac{dx_1}{dt} = \frac{\Gamma_1}{2\pi} \frac{y_2 - y_1}{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

\[ V_1 = \frac{dy_1}{dt} = \frac{-\Gamma_1}{2\pi} \frac{x_2 - x_1}{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

\[ U_2 = \frac{dx_2}{dt} = \frac{\Gamma_1}{2\pi} \frac{y_1 - y_2}{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

\[ V_2 = \frac{dy_2}{dt} = \frac{-\Gamma_1}{2\pi} \frac{x_1 - x_2}{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

$r^2 = a \text{ in } t^2 \text{ in general}$

March in time

3. Time Marching schemes (overview)

Consider one 1st-order diff. eq. \( \frac{dF}{dt} = D(t) = \text{derivative I.e.} \]

\( = \text{func. of time} \)

\( \delta \) the variables

\[ F^{(n)} \]

\[ \text{exact solution} \]

\[ t^{(n)} = n \text{th time step} \]

\[ F^{(n)} = \text{value of } F \text{ at } t^{(n)} \]

Goal: for known $F^{(n)}$, predict $F^{(n+1)}$
a. Forward time-difference scheme

\[ \frac{F^{(n+1)} - F^{(n)}}{\Delta t} = 0 \]

\[ \frac{dF}{dt} = 0 \]

\[ F^{(n+1)} = F^{(n)} + 0 \Delta t \]

- Not good unless \( \Delta t \) is very small
- Can be unstable if \( \Delta t \) is not small enough
- This is an explicit scheme
- 1st order accurate in time

b. Backward time-difference scheme

\[ D = \text{slope at } t^{(n+1)} \]
This scheme is **implicit** since we don't know \( O^{(n+1)} \) in \( \text{eqn} \)

- More algebra
- Still 1st-order accurate in time
- Unconditionally stable for any \( \Delta t \).

**Central difference scheme**

Evaluate derivatives @ \( t(t^{n+1/2}) \)

\[
D^{(n+1/2)} = \frac{D^{(n)} + D^{(n+1)}}{2} \quad \Rightarrow \quad F^{(n+1)} = F^{(n)} + D \cdot \Delta t
\]

- 2nd-order accuracy in time (better than the first 2 \( \alpha/\Delta t \))
- Implicit since we need \( O^{(n+1)} \)
- Stable for large \( \Delta t \), but not unconditionally stable
1. Predictor-corrector time marching scheme

- Use forward diff to predict $F_{\text{pred}}^{(n+1)}$.
- Then calculate $O_{\text{pred}}^{(n+1)} @ F_{\text{pred}}^{(n+1)}$.
- Then calculate the avg. @ $t^{(n+1/2)}$, $O^{(n+1/2)} = (O^{(n)} + O_{\text{pred}}^{(n+1)}) / 2$.
- Use central difference to correct $F \Rightarrow$ get $F_{\text{corr}}^{(n+1)}$.

- $2^{nd}$-order accurate in time.
- Stable for large $\Delta t$ [can be unstable if $\Delta t$ too large].
- Explicit scheme.
c. Runge-Kutta time marching scheme  4th-order accurate

Step 1: Calculate \( D_1 = \left[ \text{slope at } t^{(n)} \right] \) \( e \Delta t \) to \( t^{(n+\frac{1}{2})} \)

Step 2: \( D_2 = \left[ \text{slope at } t^{(n+\frac{1}{2})} \right] \) \( e \Delta t \) to \( t^{(n+1)} \)

Step 3: \( D_3 = \left[ \text{slope at } t^{(n+1)} \right] \) \( e \Delta t \) to \( t^{(n+\frac{3}{2})} \)

Step 4: Calculate \( D_4 = \left[ \text{slope at } t^{(n+\frac{3}{2})} \right] \) \( e \Delta t \) to \( t^{(n+1)} \)

\[ F^{(n+1)} = F^{(n)} + \Delta t \left( \frac{D_1}{6} + \frac{D_2}{3} + \frac{D_3}{3} + \frac{D_4}{6} \right) \]

\[ + O(\Delta t^5) \]

- Explicit
- 4th-order i.e. very stable but not unconditionally stable
- Can use a much larger \( \Delta t \) than the previous method
Here are variables $x_1, y_1, x_2, y_2$

Order

\[
\frac{dx_1}{dt} = \_ \quad \frac{dy_1}{dt} = \_
\]

Vector

\[
\text{position} = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix}
\]

\[
\mathbf{v}(t, \text{vinit}) = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dy_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dy_2}{dt} \end{bmatrix}
\]

Proceed in three ways: R-K.

(See HW 6)

You will do this for 2 vortices of arbitrary

strength & arbitrary initial locations