Today, we will:

- Derive the vorticity equation, and discuss its physical meaning, significance, and usefulness.

**E. The Vorticity Equation**

1. Derivation

   - N-S eq. for incompressible flow with constant properties

   \[
   \rho \frac{du_i}{dt} + \rho \left( u_j \frac{du_i}{dx_j} \right) = -\frac{\partial p}{\partial x_i} + \rho g_i + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j}
   \]

   Use same vector identity we used for Boussinesq:

   \[
   \frac{\partial}{\partial x_i} \left( \frac{1}{2} u_i u_j \right) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) u_i u_j
   \]

   \[
   \rho \frac{du_i}{dt} + \rho \left( \frac{1}{2} u_j u_j \right) = -\frac{\partial p}{\partial x_i} - \rho \frac{2}{\partial x_i} (\Theta t) + \rho u_j \frac{\partial u_i}{\partial x_j} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j}
   \]

   Take curl of the whole eq. — \[ \varepsilon_{kmi} \frac{2}{dx_m} \left[ \varepsilon_{kmi} \frac{2}{dx_m} \right] \]

   \[
   \rho \left[ \varepsilon_{kmi} \frac{2}{dx_m} \frac{du_i}{dx_i} + \varepsilon_{kmi} \frac{2}{dx_m} \left( \frac{1}{2} u_j u_j \right) - \varepsilon_{kmi} \frac{2}{dx_m} \left( u_j \varepsilon_{ijn} \omega_n \right) \right]
   \]

   \[
   = -\varepsilon_{kmi} \frac{2}{dx_m} \frac{2}{dx_i} (\Theta t) - \rho \varepsilon_{kmi} \frac{2}{dx_m} \frac{2}{dx_i} (\omega_n) + \mu \varepsilon_{kmi} \frac{\partial^3 u_i}{\partial x_m \partial x_i \partial x_j}
   \]

   \[
   \rho \frac{2}{dx_m} \left( \varepsilon_{kmi} \frac{du_i}{dx_m} \right) \rightarrow \varepsilon_{kmi} \nu \text{ not a func. of } \nu\
   \]

   \[
   0 = \rho \frac{2}{dx_m} \left( \varepsilon_{kmi} \frac{du_i}{dx_m} \right)
   \]
2: \[ \rho \varepsilon_{kmi} \frac{2}{dx_m} \frac{2}{dx_i} (\varepsilon u_{ij}) \]

Assume this is smooth, continuous, therefore of space, order of diff. does not matter.

\[ \varepsilon_{kmi} = 0 \]

3: Save for last.

4: \[ 0 \]

5: \[ 0 \]

For some reason if \( \varepsilon_{kmi} = 0 \)

\[ \varepsilon_{kmi} \text{ is not a hint of spice} \]

6: \[ M \varepsilon_{kmi} \frac{2}{dx_m} \frac{2}{dx_j} (\varepsilon u_{ij}) = M \frac{2}{dx_j} \frac{2}{dx_j} (\varepsilon_{kmi} \frac{2}{dx_m}) \]

\[ \varepsilon_{kmi} \text{ is not a hint of spice} \]

\[ \omega_k \]

7: \[ -\rho \varepsilon_{kmi} \frac{2}{dx_m} (u_j \varepsilon_{ijm} \omega_n) = -\rho \varepsilon_{kmi} \varepsilon_{ijm} \frac{2}{dx_m} (u_j \omega_n) \]

\[ \varepsilon - \varepsilon \text{ relationship} \Rightarrow \varepsilon_{kmi} \varepsilon_{ijm} = \delta_{kj} \delta_{mn} - \delta_{km} \delta_{mj} \]

\[ \delta_{kj} \delta_{mn} \frac{2}{dx_m} (u_j \omega_n) - \delta_{km} \delta_{mj} \frac{2}{dx_j} (u_j \omega_k) \]

\[ = -\rho \left[ \frac{2}{dx_n} (u_k \omega_n) - \frac{2}{dx_j} (u_j \omega_k) \right] \]

\[ = -\rho \left[ \frac{2}{dx_j} (u_k \omega_j) - \frac{2}{dx_j} (u_j \omega_k) \right] \]
Product rule:

\[ 3 = -\rho \left[ u_k \frac{\partial u_j}{\partial x_j} + u_j \frac{\partial u_k}{\partial x_j} - \omega_k \frac{\partial \omega_j}{\partial x_j} - \omega_j \frac{\partial \omega_k}{\partial x_j} \right] \]

\[ \text{(incomp.)} \]

\[ u_k \frac{\partial u_i}{\partial x_j} = u_k \frac{\partial}{\partial x_j} \varepsilon_{jmn} \frac{\partial u_m}{\partial x_n} = 0 \]

\[ \text{symm. in } j \text{, } \text{anti-symm. in } j \text{, } i \text{, } m \]

\[ 3 = -\rho \omega_j \frac{\partial u_k}{\partial x_j} + \rho u_j \frac{\partial \omega_k}{\partial x_j} \]

Put all these back into Eq (1):

\[ \rho \left[ \frac{\partial u_k}{\partial t} + u_j \frac{\partial u_k}{\partial x_j} \right] = \rho \omega_j \frac{\partial u_k}{\partial x_j} + \nu \frac{\partial^2 u_k}{\partial x_j \partial x_j} \]

\[ \frac{\partial u_k}{\partial t} \quad \text{(material derivative)} \]

\[ \therefore \rho \text{ conserve } \omega, \text{ note that } \nu = \frac{\mu}{\rho} = \text{kinematic viscosity} \]

\[ \frac{\partial u_k}{\partial t} = \omega_j \frac{\partial u_k}{\partial x_j} + \frac{\nu}{\partial x_j \partial x_j} \]

\[ \text{III} \]

\[ \frac{\partial \omega}{\partial t} = (\omega \cdot \nabla) \vec{u} + \nu \nabla^2 \vec{\omega} \]

\[ \text{(2)} \]
2. Physical meaning, significance of the vorticity $\mathbf{\omega}$:

\[
\frac{D\mathbf{\omega}}{Dt} = \mathbf{\omega} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{\omega}
\]

- Rate of change of vorticity due to stretching:
  - Stretching, e.g., a diffuser
  - Analogy: Ice skater

- Rate of change of vorticity due to visco diffusion:
  - Vortex lines:
  - Force on spins (conserve angular momentum)

- Tilting:
  - E.g., a velocity gradient

We generated vorticity in the $x$-direction where originally there was no $x$ component.

[This mechanism generates 3-D vorticity, especially in turbulent flow]
Term III: Viscous diffusion term

\[ \nu \frac{2w_w}{d_x} \frac{dx}{2y} \]

Vorticity diffuses radially outward with time due to viscosity.

E.g. at \( t = 0 \) we start with a line vortex

\[ U_0 = \frac{\Gamma}{2\pi r} \]

The inner region is called the viscous core.

[\( \omega = \text{constant} \)]

Very inner portion is like solid body rotation.

\[ U_0 = \frac{\Gamma}{2\pi r} \] outside the core (outer region)

[\( \omega = 0 \)]

\[ \Gamma = \oint_C \mathbf{u} \cdot d\mathbf{s} = \iint \mathbf{\omega} \cdot d\mathbf{A} = \text{constant} \]

provided that curve \( C \) is outside the core region.

Note: This discussion was added after class since we ran out of time.

[Vorticity is diffused, but the net strength (circulation) of the vortex remains constant.]

\[ \text{The area under any of the above vorticity curves remains constant.} \]