Today, we will:

- Discuss the “dynamic pressure” – also called the “modified pressure”
- Do an example analytical solution – steady flow between infinite parallel plates
- If time, briefly discuss some other exact analytical solutions

2. The “dynamic pressure” (or better “modified pressure”)

\[ \rho \frac{D\vec{u}}{Dt} = \rho \vec{g} - \nabla \psi + \mu \nabla^2 \vec{u} \]  

Hydrostatic condition (flow is at rest \( \rightarrow \) no flow \( \rightarrow \) \( \vec{u} = 0 \))

1. becomes \( 0 = \rho \vec{g} - \nabla \psi \)  

2. \( \psi \) = hydrostatic pressure

Subtract (2) from (1)

\[ \rho \frac{D\vec{u}}{Dt} = -\nabla (\psi - \psi_s) + \mu \nabla^2 \vec{u} \]

\( \psi_s \) = Kutta calls this the "dynamic pressure"

Poor choice of units \( \rightarrow \) dynamic pressure \( \equiv \frac{1}{2} \rho \vec{u}^2 \) or \( \frac{1}{2} \rho \vec{V}^2 \)

Other names are “modified pressure” \& “excess pressure”\& “total hydrostatic pressure”

Gravity term disappear in the NS eq.

E.g., if \( \vec{g} \) acts in \(-z\) direction, \( \psi_s = \bar{\psi} = \psi + \rho g z \)

[Known then drop the subscript \( \bar{\phi} \) from here on]
Physically this means:

Gravity does not affect the dynamic of the flow field

- It only contributes to a "superposed" hydrostatic pressure component.

[In CFD code like Fluent, \( p \) is actually the modified pressure]

Exception: if there are free surfaces

or interfaces between two fluids

- If there are buoyancy forces

  e.g. Boussinesq approx \( \rightarrow \) add density buoyancy effects in the gravity term

\[
\rho \frac{D\mathbf{u}}{Dt} = -\nabla p_d + \mu \nabla^2 \mathbf{u}
\]  

Notice: gravity term is gone.

Procedure: - Solve (3) for \( \mathbf{u} \) i.e. \( p_d \)

  * Add hydrostatic component back in at the end
B. Examples of Exact Laminar Flow Solutions

1. Steady flow between infinite parallel plates

**Given:** A fluid flows between two walls as sketched. The following assumptions and approximations are made:

1. walls are infinitely long, horizontal, and parallel.
2. upper wall moves at \( u = U \)
3. lower wall is stationary: \( u = 0 \)
4. incompressible flow with constant properties
5. imposed pressure gradient \( \frac{\partial p}{\partial x} \) (typically negative)
6. flow is steady: \( \frac{\partial (\text{anything})}{\partial t} = 0 \)
7. flow is two-dimensional into the page: \( \frac{\partial (\text{anything})}{\partial z} = 0 \) and \( w = 0 \)
8. gravity acts in the \(-y\) direction; \( \vec{g} = (0, -g, 0) \)
9. flow is fully developed: \( \frac{\partial (\text{any velocity})}{\partial x} = 0 \) (velocity profile does not change in \( x \))

**To do:** Calculate the velocity and pressure fields.

**Solution:**

\[ \frac{\partial}{\partial x} = 0 \quad \Rightarrow \quad \nabla \neq \text{func} \left( y \right) \]

\[ \left( 2 \right) \quad \left( 7 \right) \]

\[ \begin{align*}
\text{Continuity:} & \\
\frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial u}{\partial z} & = 0
\end{align*} \]

\[ \Rightarrow \quad \nabla \neq \text{func} \left( y \right) \quad \text{We know that} \quad \nabla \neq \text{func} \left( x \right) \quad \text{(6)} \]

\[ \nabla \neq \text{func} \left( x \right) \quad \text{(9)} \]

\[ \nabla \neq \text{func} \left( x \right) \quad \text{(3)} \]

\[ \begin{align*}
\text{Boundary condition:} & \\
v & = 0, \quad y = 0
\end{align*} \]

\[ \Rightarrow \quad v = \text{constant} \quad \text{everywhere} \]

\[ \begin{align*}
\text{Energy equation:} & \\
p \frac{\partial v}{\partial t} & = -\frac{\partial p}{\partial y} + \rho g \frac{\partial y}{\partial y} + \mu \nabla^2 y
\end{align*} \]

\[ v = 0 \quad \text{(Cont)} \]

\[ g_y = -g \quad \text{(Cont)} \]

\[ \frac{\partial p}{\partial y} = -\rho g \quad \text{(1)} \]

\[ \int_{\text{Int.}}: \quad p = p(x, y) = -\rho gy + f(x) \]
\[ p \left( \frac{du}{dt} + u \frac{du}{dx} + v \frac{du}{dy} + w \frac{dw}{dz} \right) = -\frac{dp}{dt} + \rho g_x + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \]

- Fully dev. cont Z-D
- \( p_x = 0 \) fully dev.
- 2-D

**But** \( u \neq \text{func. of } t, x, z \Rightarrow U = \text{func.}(y) \) only

Can we \( t \) instead of \( t \\

\[ x - \text{mom} \]

\[ \frac{dp}{dx} = \mu \frac{d^2 u}{dy^2} \]  \hspace{1cm} (2)

Recall (1) - \( \rho = -\rho gy + f(x) \)

\[ \frac{dp}{dx} = 0 + f'(x) \]

\[ \frac{dp}{dx} = \text{func. of } x \] only

\[ \frac{du}{dy} = \text{func. of } y \] only

**Eq. (2)**

\[ \frac{dp}{dx} = \mu \frac{d^2 u}{dy^2} \]

\[ \text{Func. of } x = \text{Func. of } y = \text{Constant} \]

**OR**

Alternatively, take \( \frac{d}{dx} \) of Eq. (2)

\[ \frac{d}{dx} \left( \frac{dp}{dx} \right) = \frac{d}{dx} \left( \mu \frac{d^2 u}{dy^2} \right) = 0 \]

\[ \frac{d}{dx} \left( \frac{dp}{dx} \right) = 0 \Rightarrow \frac{dp}{dx} = \text{constant} \]

\[ p \text{ at most a linear function of } x \]
Back to Eq. (1):

\[ p = -\rho g y + f(x) \]

But \( f'(x) = \frac{df}{dx} = \text{constant} \)

Integrate \( f(x) = x \frac{df}{dx} + \text{constant} \)

Finally, (i) becomes

\[ p = p(x, y) = -\rho g y + x \frac{df}{dx} + \text{constant} \]

This constant does not affect the flowfield at all except to add a constant pressure gradient everywhere.

\[ \frac{df}{dx} \]

is the imposed pressure gradient.

\[ \frac{df}{dx} = \text{constant} \]

\[ U = \frac{1}{2} \frac{df}{dx} y^2 + C_1 y + C_2 \]

\[ \frac{dU}{dy} = \frac{1}{\mu} \frac{df}{dx} y + C_1 \]

Integrate

\[ \frac{dU}{dy} = \frac{1}{\mu} \frac{df}{dx} y + C_1 \]

Integrate

\[ U = \frac{1}{2\mu} \frac{df}{dx} y^2 + C_1 y + C_2 \]
Apply BCs to get constants $c_1$ and $c_2$.

@ $y=0, u=0$ \hspace{1cm} $0 = 0 + 0 + c_2 \rightarrow c_2 = 0$

@ $y=2b, u=U$ \hspace{1cm} $U = \frac{1}{2\mu} \frac{\partial p}{\partial x} + 2c_1b$

Solve for $c_1$ \hspace{1cm} $c_1 = \frac{U}{2b} - \frac{b}{\mu} \frac{\partial p}{\partial x}$

So finally,

$$u = \frac{yU}{2b} - \frac{y}{2\mu} \frac{\partial p}{\partial x} (2b-y)$$ (4)

**Non-dimensionalize:** Let\[
\begin{align*}
U^* &= \frac{u}{U} \\
y^* &= \frac{y}{2b}
\end{align*}
\]

Plug into (4)

get\[
U^* = y^* + \beta y^* (1-y^*)
\]

where\[
\beta = \frac{-2b^2}{\mu U} \frac{\partial p}{\partial x}
\]

= pressure gradient parameter

A kind of superposition of Couette flow \to $\frac{\partial p}{\partial x} = 0 \Rightarrow U \neq 0$

* Plane Poiseuille flow \to $U = 0, \frac{\partial p}{\partial x} \neq 0$

A combination of

\[\begin{align*}
\text{left} + \text{right}
\end{align*}\]