Today, we will:

- Continue working on the example problem – steady flow between parallel plates
- Start discussing similarity solutions
- Do an example problem – Stokes’ first problem

From last lecture: the velocity and pressure solutions, found from continuity and momentum:

\[ v = 0 \]
\[ u = u(y) = \frac{yU}{2b} - \frac{y}{2\mu} \frac{\partial p}{\partial x} (2b - y) \]
\[ p = p(x, y) = -\rho gy + x \frac{\partial p}{\partial x} + \text{constant} \]

Now let’s calculate the temperature field, \( T = T(x, y) \).

\[ \rho C_p \frac{\partial T}{\partial t} = k \nabla^2 T + \phi \]

\[ \phi = 2\mu \epsilon_{ij} e_{ij} \quad \text{for incomp. flow. Since } u = u(y), \quad v = 0; \quad w = 0 \]

Only non-zero comp's of \( e_{ij} \) are \( e_{12} \) & \( e_{21} \)

\[ e_{ij} = \begin{bmatrix} 0 & \frac{1}{2} \frac{\partial u}{\partial y} & 0 \\ \frac{1}{2} \frac{\partial u}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

\[ \phi = 2\mu \left[ e_{12}^2 + e_{21}^2 \right] = 4\mu \left( \frac{1}{2} \frac{\partial u}{\partial y} \right)^2 \]

En. eq. reduces to

\[ 0 = k \frac{d^2 T}{dy^2} + M \left( \frac{du}{dy} \right)^2 \quad (5) \]
Consider plane Couette flow \((\frac{\partial U}{\partial x} = 0)\)

\[
U(y) = \frac{yU}{2b} \quad \Rightarrow \quad \frac{\partial U}{\partial y} = \frac{U}{2b}
\]

\[(5) \Rightarrow \]

\[
0 = k \frac{1}{y^2} \frac{\partial^2 T}{\partial y^2} + \frac{U^2}{y^2}
\]

\[
\frac{1}{y^2} \frac{\partial^2 T}{\partial y^2} = \left( -\frac{U^2}{k y^2} \right) \quad \Rightarrow \quad \text{integrate twice}
\]

\[
T = \frac{-U}{8k} \frac{y^3}{b^2} + c_1 y + c_2
\]

BC: on \(T \to 0\) at \(y = 0\), \(c_1 = c_2 = 0\)

\[\begin{align*}
\text{at } y = 0, & \quad T = T_0 \quad \Rightarrow \quad c_2 = T_0 \\
\text{at } y = 2b, & \quad T = T_i \quad \Rightarrow \quad \text{apply } \quad c_1 = \frac{T_i - T_0}{2b} + \frac{MU^2}{8kb} \\
\text{linear in } y & \quad \text{parabolic in } y
\end{align*}\]

\[
T = T_0 + \left( T_i - T_0 \right) \frac{y}{2b} + \frac{MU^2}{8k} \left[ \frac{y}{2b} - \left( \frac{y}{2b} \right)^2 \right]
\]

\[\text{represent conduction} \begin{cases}
\text{even if there were}
\text{no flow at all}
\end{cases} \quad \text{coming from}
\]

\[
T_i \quad \text{at} \quad T_0
\]

\text{combustion effect of conduction & wire heat addition (actual profile)}
Nondimensionalize:

\[
\begin{align*}
    & y^* = \frac{y}{2b} \\
    & T^* = \frac{T - T_0}{T_i - T_0}
\end{align*}
\]

Let \( y^* = 1 \) \( T^* = 1 \) \( y^* = 0 \) \( T^* = 0 \)

Plug into (6)

\[\begin{align*}
    T^* &= y^* + \frac{MU^2}{2k(T_i - T_0)}(y^* - y^*^2)
\end{align*}\]

\( \frac{dy}{dx} = 0 \) (Courant flow)

\( Br = \) Brinkman #

Temp profile shape depends on \( Br \)

\[
Br = \frac{\text{Volumetric diffusion}}{\text{Conduction}}
\]

Example:

Let \( T_i - T_0 = 10 \degree C \) \( U = 10 \text{ m/s} \)

\[
\begin{align*}
    & \text{Water} \to M = 1.0 \times 10^{-3} \text{ kg/m}^3 \\
    & k = 0.60 \text{ W/m} \cdot \text{C}
\end{align*}
\]

\( Br = 0.017 \) (small)

\( T \) will be nearly linear

\[
\begin{align*}
    & \text{Oil} \to SAE 30 \text{ motor oil} \\
    & M = 0.29 \text{ kg/m}^3 \\
    & k = 0.145 \text{ W/m} \cdot \text{C}
\end{align*}
\]

\( Br = 20 \) (large)

\( T \) will be dominated by the \text{Villary diffusion}
Air: \( \mu = 1.8 \times 10^{-5} \text{ kg/m/s} \),
\[ k = 0.26 \text{ W/m K} \]

\( \beta_p = 7 \times 10^{-4} \leq 1 \quad \text{VERY SMALL!} \)

\[ \gamma \]  \text{Volume thin film is negligible}

For an ideal gas at low Mach #, we ignored \( \phi \) in the energy eq.

\[ \frac{DT}{Dt} = K \frac{\partial^2 T}{\partial x_1 \partial x_1} \left( \phi \right) \quad (K = \frac{k}{\rho C_p} = thermal~diffusivity) \]

This flow (air) is most likely turbulent \( \implies \) so our solution may not be accurate

\[ \frac{\partial}{\partial x_1} \left( \frac{1}{\rho} \frac{\partial P}{\partial x_1} \right) = \mu \frac{\partial^2 u}{\partial x_1^2} \]

All of this assumes that \( \Delta T \) is small

\[ \downarrow \]

We assume \( k, \mu, C_p = \text{cont} \).

If \( \Delta T \) is not small, the \( T \) must appear in mom. eq.

\( \text{(due to } \mu \text{ and } \rho) \)

Mom, cont. 5. \( \text{En. will all be coupled.} \)

\[ \begin{align*}
\text{8. Exact Laminar Flow Solutions} \\
& - 1. \text{ Flow between parallel plates}
\end{align*} \]

2. Steady pipe flow

3. Flow between concentric cylinders
4. Axial flow in an annulus

\[ \frac{dp}{dx} = 0 \]

5. Cylinder rotating in an infinite fluid

6. Falling film

7. Other - There are a few other

For more complicated flows, we use:

- Similarity solutions
- \( CF_0 \)
- Approximations
  (e.g. Potential, RLS)