Today, we will:

- Finish discussing the 2-D stagnation point similarity problem
- Begin a discussion of creeping flow

From last lecture:

4. 2-D Stagnation Point Flow
Let \( \psi = BxF(y) \) in the boundary layer.
Thus, \( u = \frac{\partial \psi}{\partial y} = BxF'(y) \) and \( v = \frac{\partial \psi}{\partial x} = BF(y) \).
Now plug these expressions into the \( x \)- and \( y \)-momentum equations to generate the similarity solution.

\[
\begin{align*}
x \text{-mom:} & \quad \frac{du}{dx} + \nu \frac{dv}{dy} = \frac{1}{\rho} \frac{dp}{dx} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\
(\beta x F')(\beta F') + (-B)(B x F'') &= -\frac{1}{\rho} \frac{dp}{dx} + \nu (\partial F''')
\end{align*}
\]

\[
\begin{align*}
x \text{-mom:} & \quad BF'' = -\frac{1}{\rho \beta} \frac{dp}{dx} - \nu F'' \\
y \text{-mom:} & \quad BF' = \frac{1}{\rho \beta} \frac{dp}{dy} - \nu F'
\end{align*}
\]

"Trick" to get rid of pressure \( \rightarrow \) cross differentiate.
Take \( \frac{2}{\beta} \) of (5) \( \rightarrow \) take \( \frac{3}{\beta} \) of (4)

\[
\begin{align*}
\frac{2}{\beta} \text{ of } \text{Eq. (5)} & \quad \frac{2}{\beta} \left( \frac{dp}{dx} \right) = 0 \\
\text{or} \quad \frac{2}{\beta} \left( \frac{dp}{dx} \right) = 0 & \quad \text{since } \rho = \text{smooth continuity} \text{ fun. of } xy
\end{align*}
\]

\[
\begin{align*}
\frac{2}{\beta} \text{ of } \text{Eq. (4)} & \quad \frac{2}{\beta} \left[ F'' + \frac{\beta}{\nu} FF'' - \frac{\beta}{\nu} F' \right] = \frac{1}{\rho \beta x} \frac{dp}{dy} \left( \frac{dp}{dx} \right) = 0 \\
\text{fun. of } y
\end{align*}
\]

Integrate w.r.t. \( y \) \( \Rightarrow \quad F'' + \frac{\beta}{\nu} FF'' - \frac{\beta}{\nu} F' \left( \frac{F'}{F} \right)^2 = C_1, \quad (6)\)
Apply a BC to get $C_i$ \[ @ y \to \infty, F \to y \]
\[ \therefore F' \to 1 \]
\[ F'' \to 0 \]
\[ F''' \to 0 \]

Eq (6) becomes \[ \frac{B}{\nu} (1)^3 = C_i \to C_i = \frac{B}{\nu} \] \[ @ y \to \infty \]

\[ \therefore (7) \to \]
\[ F''' + \frac{B}{\nu} (FF'' - F'^2 + 1) = 0 \]

$x$ has disappeared from the problem! We have reduced the # of independent variables by one $(x,y) \to y$

**Check BC's:**

\[ @ y = 0 \text{ no slip} \to u = 0, v = 0 \text{ \ for all } x \]
\[ u = Bx F'(y) \]
\[ \to \quad F'(0) = 0 \]
\[ v = -B F(y) \]
\[ \to \quad F(0) = 0 \]
\[ @ y \to \infty, \quad u \to Bx \]
\[ \to \quad F'(\infty) = 1 \]
\[ @ y \to \infty, \quad v \to -By \]
\[ F(y) \to y \quad \text{as} \quad y \to \infty \]

**Similarity has been achieved**

- Non-dimensionalize before solving

Need to create a length scale \[ \{L\} = \{\frac{l^2}{\ell}\} \quad \{B\} = \{\frac{-1}{\ell}\} \]

Let length scale \[ = \sqrt{\frac{\nu}{B}} \]

Let \[ \eta = \frac{y}{\sqrt{\nu/B}} \]

\[ \{F\} = \{L\} \to \text{ Let } f(\eta) = \frac{F(y)}{\sqrt{\nu/B}} \]

Plug $\eta$ into Eq (7) \[ \text{ algebra} \]
\[ f'' + ff' + 1 - f'^2 = 0 \]  \hspace{2cm} (8)

with BCs:
\[
\begin{align*}
    f(0) &= 0 \\
    f'(0) &= 0 \\
    f'(\infty) &= 1
\end{align*}
\]

How to solve? Ode, but nonlinear

No analytical soln. — but perfect problem for Runge-Kutta technique

3\textsuperscript{rd}-order ode. \to break up into 3 1\textsuperscript{st}-order odes

\[
\begin{align*}
    f'' &= (f'')' = f'^2 - 1 - ff'' \\
    (f')' &= f'' \\
    (f)' &= f'
\end{align*}
\]

Let
\[
\begin{align*}
    Y_1 &= f'' \\
    Y_2 &= f' \\
    Y_3 &= f
\end{align*}
\]

Define derivatives
\[
\begin{align*}
    D_1 &= Y_1' = (f'')' = f'^2 - 1 - ff'' \\
    D_2 &= Y_2' = f'' = Y_1 \\
    D_3 &= Y_3' = f' = Y_2
\end{align*}
\]

\[
\begin{align*}
    D_1 &= Y_2^2 - 1 - Y_3 Y_1 \\
    D_2 &= Y_1 \\
    D_3 &= Y_2
\end{align*}
\]

BC's:
\[
\begin{align*}
    f(0) &= 0 \rightarrow Y_3(0) = 0 \\
    f'(0) &= 0 \rightarrow Y_2(0) = 0 \\
    f'(\infty) &= 1 \rightarrow Y_2(\infty) = 1
\end{align*}
\]

For R-K to work, need BC, e.g. \( \eta = 0 \) for all \( Y \)'s

But we don't know \( Y_1(0) \)

Let's guess \( Y_1(0) \) and iterate until \( Y_2(\infty) = 1 \). Only one value of \( Y_1(0) \) will produce \( Y_2(\infty) = 1 \).
I use Method -- see website (file is poss.)

Note: \( f \propto -v \)
\[ f' = u \]
\[ f'' = \frac{du}{dy} \propto 2 \text{ (char v reg)} \]

\[ u \propto y \]
\[ f', v, n \]

velocity profile that is our similarity solution

D. Creeping Flow (also Stokes flow, low Reynolds & flow)

Assumption/Approx. -- incompressible \((p = \text{const}, \mu = \text{const})\)
- ignore gravity
- viscosity-dominated flow

\[ \text{Eqn: cont.:} \]
\[ \frac{du_i}{dx_i} = 0 \]

\[ \text{Mom.:} \]
\[ P \left( \frac{2u_i}{2t} + u_k \frac{du_i}{dx_k} \right) = -\frac{2P}{2x_i} + \mu \frac{2^2 u_i}{2x_k 2x_k} \]

A. Non-dimensionalization of the eqn's

We also have fluid properties \( \mu, \rho \)

We also need characteristic time scale & length scale
Time → create a time scale → \( \frac{L}{U} = \text{Charndavhi time scale} \)

Pressure → "pressure" two choices \( \rho U^2 \)

or \( \frac{MU}{L} \)

For High Re application (e.g., air flow) → pick \( \rho U^2 \)

(Reynolds effects dominate)

For Low Re application,

Viscosity effects dominate Reynolds effects, so choose

\( \frac{MU}{L} = \text{Charndavhi pressure scale} \)

\( \star \) THIS PRESSURE SCALE IS MORE APPROPRIATE THAN \( \rho U^2 \) FOR LOW REYNOLDS \& PLANS

Next class — plug all these into the eq of motion