Benard Convection Problem - Eigenvalue Solution for Marginal Stability

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First define the roots q0, q, and qs as functions of wavenumber K and Rayleigh number Ra:

$$q0(K,Ra) := K\cdot\sqrt{\left(\frac{Ra}{K^4}\right)^{\frac{1}{3}} - 1} \qquad q(K,Ra) := \sqrt{K^2\cdot\left[1 + \frac{1}{2}\cdot\left(\frac{Ra}{K^4}\right)^{\frac{1}{3}}\cdot\left(1 + \sqrt{3}\cdot i\right)\right]} \qquad qs(K,Ra) := \sqrt{K^2\cdot\left[1 + \frac{1}{2}\cdot\left(\frac{Ra}{K^4}\right)^{\frac{1}{3}}\cdot\left(1 - \sqrt{3}\cdot i\right)\right]}$$

Now define the matrix as given in the class notes (Matrix M is also a function of K and Ra):

$$M(K,Ra) := \begin{bmatrix} \cos\left(\frac{q0(K,Ra)}{2}\right) & \cosh\left(\frac{q(K,Ra)}{2}\right) & \cosh\left(\frac{qs(K,Ra)}{2}\right) \\ -q0(K,Ra)\cdot\sin\left(\frac{q0(K,Ra)}{2}\right) & q(K,Ra)\cdot\sinh\left(\frac{q(K,Ra)}{2}\right) & qs(K,Ra)\cdot\sinh\left(\frac{qs(K,Ra)}{2}\right) \\ \left(q0(K,Ra)^2 + K^2\right)^2\cdot\cos\left(\frac{q0(K,Ra)}{2}\right) & \left(q(K,Ra)^2 - K^2\right)^2\cdot\cosh\left(\frac{q(K,Ra)}{2}\right) & \left(qs(K,Ra)^2 - K^2\right)^2\cdot\cosh\left(\frac{qs(K,Ra)}{2}\right) \end{bmatrix}$$

Define the determinant of this matrix, and its root as functions of K and Ra. *Note*: In order to use the root function, an initial guess for Ra must be specified. I also had to lower the tolerance for the root function from the default:

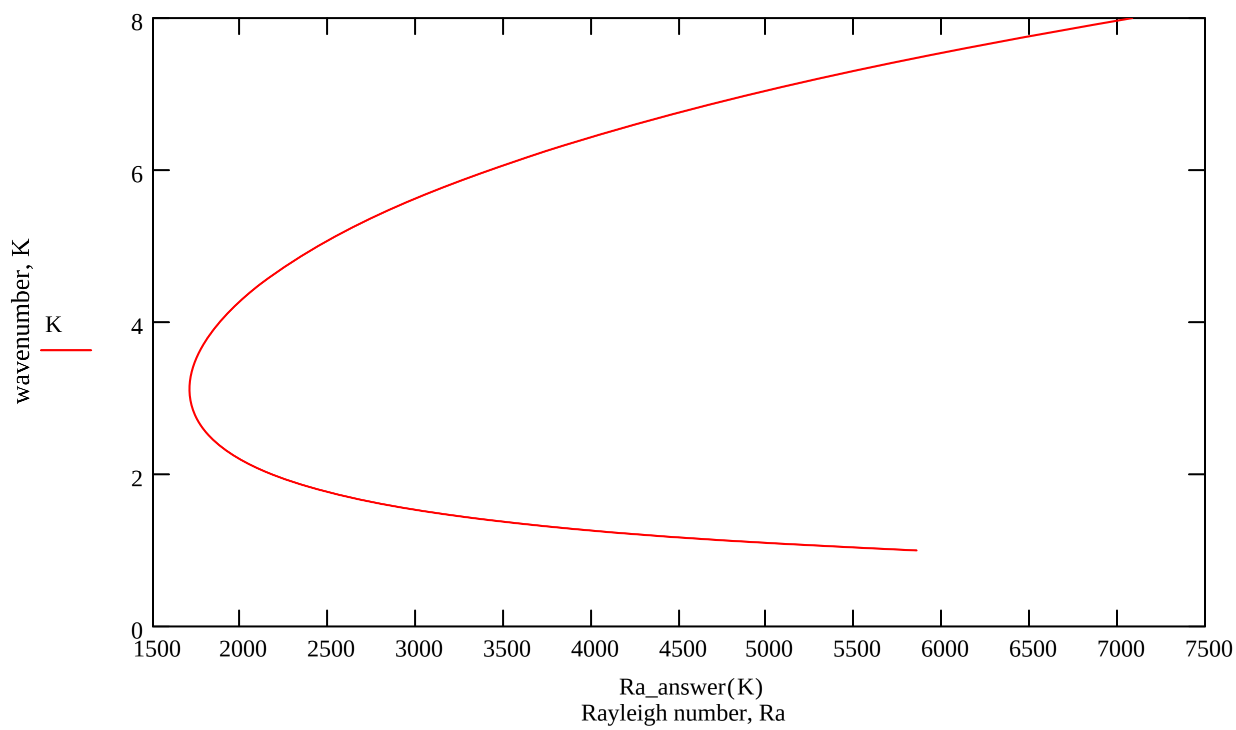
$Ra\_guess := 4000 \qquad Ra := Ra\_guess \qquad determ(K,Ra) := |M(K,Ra)| \qquad Ra\_answer(K) := root(determ(K,Ra), Ra)$

Verify for some initial K: $TOL = 1\times 10^{-3} \qquad TOL := 1.0\cdot 10^{-6} \qquad K := 1 \qquad Ra\_answer(K) = 5.854\times 10^{3}$

Specify a range of wavenumber K, and solve for the corresponding Ra which yields non-zero roots.

$$ntotal := 100 \qquad Kmin := 1 \qquad Kmax := 8 \qquad K := Kmin, Kmin + \frac{(Kmax - Kmin)}{ntotal} .. Kmax$$

Plot the marginal (neutral) stability curve:



Write the minimum Ra for stability, along with its wavenumber, K (I'm not sure why this works in Mathcad):

$K = 3.111 \qquad Ra\_answer(K) = 1.708\times 10^{3}$