The Bénard Problem – A Summary

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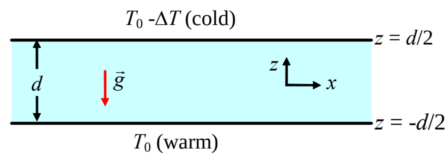
Latest revision: 08 February 2008

1. Introduction and Problem Setup



Consider two infinite, stationary, parallel flat plates separated by distance d. The lower plate is at temperature T_0 and the upper plate is at temperature T_0 - ΔT (colder). Initially the flow is *at rest*, but this is an unstable situation since the warm fluid on the bottom wants to rise. We examine this problem using linear stability analysis.

2. Summary of Linear Stability Analysis (applied to this problem):

The in-class analysis follows Kundu, Section 12.3 closely, filling in some of the details:

- **Step 0**. Start with the Boussinesq equations (Navier-Stokes equations for buoyant flows) for *total* flow variables ($\tilde{q}$):

  $\dfrac{\partial \tilde{u}_i}{\partial x_i} = 0$ (1), $\rho_0\left(\dfrac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \dfrac{\partial \tilde{u}_i}{\partial x_j}\right) = -\dfrac{\partial \tilde{p}}{\partial x_i} - \rho_0 g \delta_{i3}\left[1-\alpha\left(\tilde{T}-T_0\right)\right] + \mu \dfrac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j}$ (2), and $\dfrac{\partial \tilde{T}}{\partial t} + \tilde{u}_j \dfrac{\partial \tilde{T}}{\partial x_j} = \kappa \dfrac{\partial^2 \tilde{T}}{\partial x_j \partial x_j}$ (3).

  This represents 5 equations and 5 unknowns.
- **Step 1.** Generate the *basic state* equations (1b), (2b), and (3b). Here, we assume no flow, so that $U_i = 0$, $P = P(z)$ = hydrostatic pressure (although P does not vary with z exactly linearly), and $\overline{T}(z) = T_0 - \Gamma\left(z + \dfrac{d}{2}\right)$, where $\Gamma \equiv \Delta T / d$.
- **Step 2.** Add disturbances ($\tilde{q} = Q + q$), and plug into (1), (2), and (3): This generates *total equations* (1t), (2t), and (3t).
- **Step 3.** Subtract basic state equations from the total equations: This generates *disturbance equations* (1d), (2d), and (3d).
- **Step 4.** *Linearize* the disturbance equations to generate the *linearized disturbance equations* (1l), (2l), and (3l):

  $\dfrac{\partial u_i}{\partial x_i} = 0$ (1*l*), $\dfrac{\partial u_i}{\partial t} = -\dfrac{1}{\rho_0}\dfrac{\partial p}{\partial x_i} + g\delta_{i3}\alpha T' + \nu \dfrac{\partial^2 u_i}{\partial x_j \partial x_j}$ (2*l*), and $\dfrac{\partial T'}{\partial t} - \Gamma w = \kappa \dfrac{\partial^2 T'}{\partial x_j \partial x_j}$ (3*l*) . This still represents 5 equations and 5 unknowns (now the *disturbance* variables are the unknowns since the basic state is known).
- **Step 5.** Solve the linearized disturbance equations (1*l*), (2*l*), and (3*l*): After some algebraic manipulation, we can eliminate pressure from the equations, and rewrite the energy and z-momentum equations as follows:

  $\dfrac{\partial T'}{\partial t} - \Gamma w = \kappa \nabla^2 T'$ (3*l* vector notation), and $\dfrac{\partial}{\partial t}\left(\nabla^2 w\right) = g\alpha \nabla_H{}^2 T' + \nu \nabla^4 w$ (6), where $\nabla_H{}^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$.

  This set of two equations and two unknowns (w and T') is *uncoupled* from the other linearized disturbance equations.

  "Semi-Normalization" of the equations: Following Kundu's notation, normalize only the *independent* variables (x_i and t), but do not *rename* the variables (kind of confusing ☹). Also introduce the Prandtl number, $\mathrm{Pr} \equiv \nu / \kappa$. Equations (3*l*) and (6) become $\dfrac{\partial T'}{\partial t} - \dfrac{\Gamma d^2}{\kappa} w = \nabla^2 T'$ (7), and $\left(\dfrac{1}{\mathrm{Pr}}\dfrac{\partial}{\partial t} - \nabla^2\right)\nabla^2 w = \dfrac{g\alpha d^2}{\nu}\nabla_H{}^2 T'$ (8) .

  The **boundary conditions** for these two equations with respect to z are $w = \dfrac{\partial w}{\partial z} = T' = 0$ at $z = \pm\frac{1}{2}$, where *this* z is defined as the *original* z divided by d ($z_{\text{new}} = z_{\text{original}}/d$).

  Method of Normal Modes: Assume disturbances that are *periodic* in x and y, but *not growing or decaying* in x or y, but *may be periodic* and *may be growing or decaying* in t (*temporal* instability). Let the disturbances be of the form $w = \hat{w}(z)e^{ikx+ily+\sigma t}$ and $T' = \hat{T}(z)e^{ikx+ily+\sigma t}$, where $\hat{w}$ and $\hat{T}$ are **complex amplitudes**, and k and l are the x and y components, respectively, of wavenumber vector $\vec{K}$. For **temporal stability analysis**, both k and l must be *real*, but σ (the complex growth rate) can be *complex* (otherwise spatial instability would also be possible). Plug these disturbances into Eqs. (7) and (8) to get: $\left[\sigma - \left(D^2 - K^2\right)\right]\hat{T} = W$ (11), $\left[\dfrac{\sigma}{\mathrm{Pr}} - \left(D^2 - K^2\right)\right]\left(D^2 - K^2\right)W = -\mathrm{Ra}\cdot K^2\hat{T}$ (12),

  where $D \equiv \dfrac{d}{dz}$, Rayleigh number = $\mathrm{Ra} = \dfrac{g\alpha\Gamma d^4}{\kappa\nu}$, $K \equiv \sqrt{k^2 + l^2}$, and $W \equiv \dfrac{\Gamma d^2}{\kappa}\hat{w}$ (defined for "convenience").

  We are now down to 2 o.d.e.s and 2 unknowns!

  The **boundary conditions** for Equations (11) and (12) are $W = DW = \hat{T} = 0$ at $z = \pm\frac{1}{2}$.
- **Step 6.** Examine stability: Finally, we solve (11) and (12) for the case of ***marginal stability*** (to be done in class).