Blasius-secondary flow flat plate boundary layer similarity solution

J. M. Cimbala

The equations to solve are \( f'' + cff'' = 0 \), where prime denotes \( d/d\eta \), and \( h'' + 0.5fh' - f + 1 - (f')^2 = 0 \).

Here, let \( c = 1/2 \), following Kundu's book. \( c := 0.5 \)

The boundary conditions are \( f(0)=1 \), \( f(\infty)=1 \), \( f'(\infty)=1 \), \( h(0)=0 \), and \( h(\infty)=0 \).

Since two of these are at infinity, \( f'(0) \) and \( h'(0) \) need to be guessed until the boundary conditions at infinity are satisfied.

First define a vector \( Y \) which contains five unknowns, \( Y_1 = f' \), \( Y_2 = f' \), \( Y_3 = f \), \( Y_4 = h' \), and \( Y_5 = h \).

**ORIGIN := 1**

<table>
<thead>
<tr>
<th>Known boundary conditions:</th>
<th>Guessed boundary conditions:</th>
</tr>
</thead>
</table>
| \( Y_2 := 0 \) \( Y_3 := 0 \) \( Y_5 := 0 \) | \( Y_1 := 0.332057 \) \( Y_4 := 1.085973166 \)

Verify the vector:

\[
Y = \begin{pmatrix}
0.33206 \\
0 \\
0 \\
1.08597 \\
0
\end{pmatrix}
\]

Now define the derivative vector \( D \) which contains the first derivative with respect to \( \eta \) of each variable in the \( Y \) vector. This derivative vector \( D \) is needed for the Runge-Kutta solution.

Now calculate the solution as \( \eta \) marches from \( \eta_{\text{start}} \) to \( \eta_{\text{end}} \). Here \( Z \) is the solution matrix, where the first column is \( \eta \), the second column is \( Y_1 \), the third column is \( Y_2 \), the fourth column is \( Y_3 \), the fifth column is \( Y_4 \), and the last column is \( Y_5 \).

Here the function \( \text{Rkadapt} \) is used, which is similar to \( \text{rkfixed} \) except it internally uses adaptable spacing instead of fixed spacing (more accuracy where needed). It reports at fixed spacing however.

\( \eta_{\text{start}} := 0 \) \( \eta_{\text{end}} := 10 \) \( \text{num\_steps} := 2000 \) \( Z := \text{Rkadapt}(Y, \eta_{\text{start}}, \eta_{\text{end}}, \text{num\_steps}, D) \)

Top portion of \( Z \):

\[
Z = \begin{pmatrix}
1 & 0 & 0.33206 & 0 & 0 & 1.08597 & 0 \\
2 & 5 \cdot 10^{-3} & 0.33206 & 1.66028 \cdot 10^{-3} & 4.15071 \cdot 10^{-6} & 1.08097 & 5.41737 \cdot 10^{-3} \\
3 & 0.01 & 0.33206 & 3.32057 \cdot 10^{-3} & 1.66028 \cdot 10^{-5} & 1.07597 & 0.01081 \\
4 & 0.015 & 0.33206 & 4.98085 \cdot 10^{-3} & 3.73564 \cdot 10^{-5} & 1.07097 & 0.01618 \\
5 & 0.02 & 0.33206 & 6.64114 \cdot 10^{-3} & 6.64114 \cdot 10^{-5} & 1.06597 & 0.02152 \\
6 & 0.025 & 0.33206 & 8.30142 \cdot 10^{-3} & 1.03768 \cdot 10^{-4} & 1.06098 & 0.02684 \\
7 & 0.03 & 0.33206 & 9.96171 \cdot 10^{-3} & 1.49426 \cdot 10^{-4} & 1.05598 & 0.03213 \\
8 & 0.035 & 0.33206 & 0.01162 & 2.03385 \cdot 10^{-4} & 1.05098 & 0.0374 \\
9 & 0.04 & 0.33206 & 0.01328 & 2.65646 \cdot 10^{-4} & 1.04598 & 0.04264 \\
10 & 0.045 & 0.33206 & 0.01494 & 3.36208 \cdot 10^{-4} & 1.04098 & 0.04786
\end{pmatrix}
\]
Bottom portion of $Z$ (to verify BCs):

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>99</td>
<td>9.95</td>
<td>$1.03781 \times 10^{-8}$</td>
<td></td>
<td>1</td>
<td>8.22921</td>
<td>$-2.12027 \times 10^{-9}$</td>
</tr>
<tr>
<td>99</td>
<td>9.955</td>
<td>$1.01667 \times 10^{-8}$</td>
<td></td>
<td>1</td>
<td>8.23421</td>
<td>$-1.69195 \times 10^{-9}$</td>
</tr>
<tr>
<td>99</td>
<td>9.96</td>
<td>$9.95948 \times 10^{-9}$</td>
<td></td>
<td>1</td>
<td>8.23921</td>
<td>$-1.27186 \times 10^{-9}$</td>
</tr>
<tr>
<td>99</td>
<td>9.965</td>
<td>$9.75637 \times 10^{-9}$</td>
<td></td>
<td>1</td>
<td>8.24421</td>
<td>$3.59864 \times 10^{-10}$</td>
</tr>
<tr>
<td>99</td>
<td>9.97</td>
<td>$9.55729 \times 10^{-9}$</td>
<td></td>
<td>1</td>
<td>8.24921</td>
<td>$-4.558 \times 10^{-10}$</td>
</tr>
<tr>
<td>99</td>
<td>9.975</td>
<td>$9.36215 \times 10^{-9}$</td>
<td></td>
<td>1</td>
<td>8.25421</td>
<td>$3.95158 \times 10^{-11}$</td>
</tr>
<tr>
<td>99</td>
<td>9.98</td>
<td>$9.17088 \times 10^{-9}$</td>
<td></td>
<td>1</td>
<td>8.25921</td>
<td>$3.29137 \times 10^{-10}$</td>
</tr>
<tr>
<td>99</td>
<td>9.985</td>
<td>$8.9834 \times 10^{-9}$</td>
<td></td>
<td>1</td>
<td>8.26421</td>
<td>$3.10305 \times 10^{-10}$</td>
</tr>
<tr>
<td>99</td>
<td>9.99</td>
<td>$8.79965 \times 10^{-9}$</td>
<td></td>
<td>1</td>
<td>8.26921</td>
<td>$1.08413 \times 10^{-9}$</td>
</tr>
<tr>
<td>000</td>
<td>9.995</td>
<td>$8.61955 \times 10^{-9}$</td>
<td></td>
<td>1</td>
<td>8.27421</td>
<td>$1.45076 \times 10^{-9}$</td>
</tr>
<tr>
<td>000</td>
<td>10</td>
<td>$8.44303 \times 10^{-9}$</td>
<td></td>
<td>1</td>
<td>8.27921</td>
<td>$1.81033 \times 10^{-9}$</td>
</tr>
</tbody>
</table>

Now generate a plot of the similarity variables:

```plaintext
 n := 1 .. num_steps
```

Blasius BL with Secondary Flow

![Blasius BL with Secondary Flow plot](attachment:plot.png)