Example Problem: 2-D Incompressible Laminar Jet

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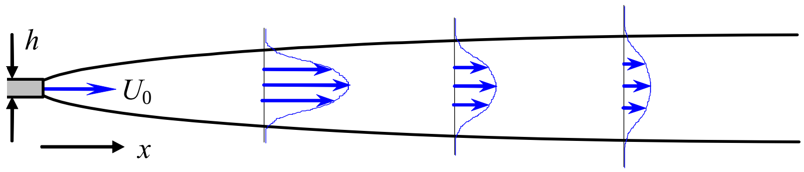
Given: A fully submerged 2-dimensional channel of height h is dumping polluted water into a large quiescent lake. Approximate the initial jet as a uniform plug flow (constant velocity U_0 over the entire height h). For simplicity, assume the properties ρ, μ, etc. of the effluent (the polluted water from the drain), are the same as those of the ambient water in the lake.



To do:

(a) Calculate the momentum flux of the jet (per unit width into the page).

Solution:

We integrate at $x = 0$, the origin of the jet, $M = \int_{-\infty}^{\infty} \rho u^2 dy = \rho \int_{-h/2}^{h/2} U_0^2 dy = \rho U_0^2 h$. In other words, since the jet is initially uniform, the momentum flux per unit width into the page is simply $\boxed{M = \rho U_0^2 h}$. ***This momentum flux is constant at any x location***, as discussed in class.

b) Calculate the initial mass flux of the jet (per unit width into the page).

Solution:

Similarly, we integrate at $x = 0$, the origin of the jet, $\dot{m}_0 = \int_{-\infty}^{\infty} \rho u dy = \rho \int_{-h/2}^{h/2} U_0 dy = \rho U_0 h$. Again, since the jet is initially uniform, the mass flux per unit width into the page is simply $\boxed{\dot{m}_0 = \rho U_0 h}$. ***This mass flux is not constant with x location***, as discussed in class – in fact, the mass flux increases downstream due to entrainment of the surrounding ambient fluid.

c) If the jet remains steady, laminar, and 2-D, the boundary layer solution discussed in class is applicable. Namely, as x increases, the jet entrains ambient fluid like $x^{1/3}$, and the polluted water becomes diluted. Assuming that at any x location the fluid inside the jet is fully mixed (well-mixed), generate an expression for how far downstream one must go in order that the jet contains only 1% of the initial effluent, and 99% ambient water.

Solution:

At any x-location, the mass flux is given by $\dot{m} = 3.302(\rho\mu M)^{1/3} x^{1/3}$. So we need to find x where $\dot{m}_0 / \dot{m} = 0.01$, or $3.302(\rho\mu M)^{1/3} x^{1/3} = 100\dot{m}_0$, or $3.302(\rho^2 \mu U_0^2 h x)^{1/3} = 100\rho U_0 h$. Solving for x gives $\boxed{x = \left(\frac{100}{3.302}\right)^3 \frac{h^2 U_0}{\nu}}$, where $\nu \equiv \mu / \rho$ is the kinematic viscosity. (*Notice that the answer is independent of density*.)

d) For $h = 5.00$ cm, $U_0 = 10.0$ cm/s, and $\nu = 0.0100$ cm^2/s, what downstream distance is required for the conditions of Part (c)?

Solution:

Plugging in the numbers, we get $\boxed{x = 6.9 \times 10^8 \text{ cm, or approximately 4300 miles}}$!

Discussion: Clearly this is *not* a realistic answer!
Why not? What have we assumed incorrectly? What would happen in real life?